INEQUALITIES AND COMPACTNESS

RADU PRECUP[1]

ABSTRACT. We use the inequalities of Hölder, Gronwall and of Wirtinger type to establish sufficient conditions for that the null function is the unique nonnegative solution of some integral inequalities. Such conditions are useful to guarantee compactness properties of Mönch and Palais-Smale type for integral operators on spaces of vector-valued functions. An application to superlinear boundary value problems in Hilbert spaces is also presented.

1. Introduction

1.1. Compactness Conditions in Nonlinear Analysis

One of the main themes of nonlinear analysis consists in establishing existence principles for operator equations of the form

$$x = N(x), \quad x \in D \subset X. \tag{1.1}$$

There are two main approaches to the theory of existence, localization and multiplicity of solutions to (1.1): Fixed point methods and variational techniques. Both of them use compactness conditions for N.

Thus, the fixed point methods usually require that N be contractive, compact, of Krasnoselskii type (i.e., sum of a contraction and a compact mapping), of Darbo type, or condensing in the sense of Sadovskii. A common generalization of all these conditions is due to Mönch [5] and requires the following implication holds:

$$\left[\begin{array}{c} \overline{C} = \overline{\text{conv}}\left(\{x_0\} \cup N(C)\right) \\ C \subset D,\ C \text{ countable} \end{array} \right\} \Longrightarrow \overline{C} \text{ compact} \Bigg]. \tag{1.2}$$

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Here X is a real Banach space, x_0 is a given point of D and *conv* denotes the convex hull. In the recent paper by O'Regan-Precup [7], it is used the following slight generalization of (1.2):

$$\text{(1.3)} \qquad \left[\begin{matrix} M = \operatorname{conv}(\{x_0\} \cup N(M)) \\ \overline{M} = \overline{C},\ C \subset M \subset D,\ C \text{ countable} \end{matrix} \right\} \Longrightarrow \overline{M} \text{ compact} \Bigg] .$$

Note that (1.3) is expressed in terms of a pair (M, C) instead of a single set C and it has been particularly useful to extend the Mönch's fixed point theorems (see [5], [3], p. 204–205 and [9]) to set-valued mappings ([7]).

Let us say that (1.1) has a variational expression if X is a real Hilbert space with inner product $(\cdot,\cdot)$ and $N = I - E'$, where I is the identity mapping of X and E is a C^1 functional on X, i.e., $E \in C^1(X;\mathbb{R})$. Clearly, if (1.1) has a variational expression, then its solutions are precisely the critical points of E, that is the zeros of E' located in D. In order to guarantee the existence (and multiplicity) of critical points, the researchers in critical point theory use the classical Palais-Smale compactness condition and variants. Thus one says [4] that E satisfies the *Palais-Smale condition* on D ((P-S)$_D$*-condition*) if

$$\left[\begin{matrix} \{x_n\} \subset D,\ E(x_n) \text{ -bounded},\ E'(x_n) \to 0 \\ \Longrightarrow \{x_n\} \text{ has a convergent subsequence} \end{matrix} \right] .$$

An interesting variant for the case when D is the ball $B_R = \{x \in X : |x| \leq R\}$ is due to Schechter [11]: We say that E satisfies *Schechter's Palais-Smale condition* on B_R ((S-P-S)$_R$*-condition*) if

$$\left[\begin{matrix} \{x_n\} \subset B_R,\ E(x_n)\text{- bounded},\ (E'(x_n), x_n) \to \nu \leq 0, \\ E'(x_n) - \frac{(E'(x_n),x_n)}{|x_n|^2} x_n \to 0 \\ \Longrightarrow \{x_n\} \text{ has a convergent subsequence} \end{matrix} \right] .$$

1.2. Abstract Existence Principles

It is well known that the compactness conditions together with suitable geometrical conditions yield existence results for (1.1). For example, we have the following fixed point theorems involving the compactness conditions of type (1.3). They are the single-valued versions of the abstract existence principles for inclusions established in [7].

Theorem 1.1. ([7]) *Let D be a closed convex subset of a real Banach space X and $N : D \to D$ be continuous. Assume that, for some $x_0 \in D$, the condition* (1.3) *holds. Then there exists at least one $x \in D$ with $x = N(x)$.*

The next result is the continuation principle accompanying Theorem 1.1:

Theorem 1.2. ([7]) *Let D be a closed convex subset of a real Banach space X, U be a relatively open subset of D and $N : \overline{U} \to D$ be continuous. Assume that, for some $x_0 \in U$, the following conditions are satisfied:*

$$\left[\left.\begin{array}{c} M \subset \operatorname{conv}(\{x_0\} \cup N(M)) \\ \overline{M} = \overline{C}, C \subset M \subset \overline{U}, C \text{ countable} \end{array}\right\} \Longrightarrow \overline{M} \text{ compact}\right], \tag{1.4}$$

$$x \neq (1-\lambda)x_0 + \lambda N(x), \quad x \in \overline{U} \setminus U, \lambda \in (0,1).$$

Then there exists at least one $x \in \overline{U}$ with $x = N(x)$.

In these two results, the geometrical conditions are : The Schauder's invariance condition $N(D) \subset D$ and the Leray-Schauder boundary condition $\lambda(N(x) - x_0) \neq x - x_0$ ($x \in \partial U = \overline{U} \setminus U$, $\lambda \in (0,1)$), respectively. The next two theorems can be seen as the analogous of Theorems 1.1-1.2 in critical point theory.

Let X be a real Hilbert space with inner product $(\cdot,\cdot)$, $D \subset X$ be a closed convex set, $E \in C^1(X;\mathbb{R})$, U be a relatively open subset of D, $x_0 \in U$ and $x_1 \in D^{ri} \setminus \overline{U}$ (here D^{ri} is the relatively algebraic interior of D). Let

$$\begin{cases} \Phi = \{\varphi \in C([0,1];D) : \varphi(0) = x_0,\ \varphi(1) = x_1\}, \\ c = \inf\limits_{\varphi \in \Phi} \max\limits_{t \in [0,1]} E(\varphi(t)),\ m = \inf\limits_{x \in D} E(x), \\ \mathcal{K}_\mu = \{x \in D : E(x) = \mu,\ E'(x) = 0\}\ (\mu \in \mathbb{R}). \end{cases}$$

Theorem 1.3. ([4]) *Assume that E satisfies the (P-S)$_D$-condition and $(I - E')(D) \subset D$.*

(a) *If* $\max\{E(x_0), E(x_1)\} \leq \inf_{x \in \partial U} E(x)$, *then* $\mathcal{K}_c \setminus \{x_0, x_1\} \neq \emptyset$.

(b) *If* $m > -\infty$, *then* $\mathcal{K}_m \neq \emptyset$. *If, in addition,* $E(x_1) \leq E(x_0)$, *then* $\mathcal{K}_m \setminus \{x_0\} \neq \emptyset$.

Theorem 1.4. ([11]) *Let $D = B_R$ Assume that E satisfies the (S-P-S)$_R$-condition. Also suppose that $-(E'(x),x) \leq C$ and $x \neq \lambda(x - E'(x))$ for all $|x| \in R$ and $\lambda \in (0,1)$. Then the statements* (a) *and* (b) *in Theorem 1.3 hold.*

1.3. Compactness Properties of Integral Operators

In applications, when N is an integral operator, the compactness conditions are most frequently fulfilled due to some particular results of compact embedding (Ascoli-Arzèla, Rellich-Kondrachov). Such results hold for several spaces of functions with values in $\mathbb{R}^n$, but fail for the corresponding spaces of functions with values in an infinite dimensional space. In such cases, the compactness conditions have to be guaranteed by extra properties of the kernel of the operator. Those properties are expressed in terms of a measure of noncompactness and require that a certain integral inequality has no other nonnegative solution than the null function. Here we present some results concerning the compactness conditions of Mönch type for the Urysohn integral operator

$$N(u)(t) = \int_0^T f(t, s, u(s))ds \quad (t \in J = [0, T]) \tag{1.5}$$

and the Volterra operator

$$N(u)(t) = \int_0^t f(t, s, u(s))ds \quad (t \in J) \tag{1.6}$$

acting on functions u from J into a real separable Banach space $(Y, |\cdot|)$. We also present our recent result on the Palais-Smale condition for the energy functional associated to the Hammerstein integral operator

$$N(u)(t) = \int_0^T k(t, s) f(s, u(s))ds \quad (t \in J) \tag{1.7}$$

acting on functions u from J into a real Hilbert space $(H, (\cdot, \cdot))$. Let $B_R = \{y \in Y : |y| \leq R\}$ and $f : J^2 \times B_R \to Y$. Denote $U = \{u \in C(J; Y) : |u(t)| < R$ for every $t \in J\}$. Let β be the *ball measure of noncompactness* defined by

$$\beta(M) = \inf\{r > 0 : M \text{ can be covered by finitely many balls of radius } r\}$$

for each bounded set M.

Theorem 1.5. ([8]) *Suppose that, for each $t \in J$, $f(t, \cdot, \cdot)$ is L^1-Carathéodory,*

$$\sup_{t \in J} \int_0^T \sup_{|x| \leq R} |f(t, s, x)| ds < \infty$$

and

$$\int_0^T \sup_{|x| \leq R} |f(t, s, x) - f(t', s, x)| \, ds \to 0 \quad \text{as } t' \to t.$$

In addition, suppose that there exists $w : J^2 \times [0, R] \to \mathbb{R}$ such that, for each $t \in J$, $w(t, \cdot, \cdot)$ is L^1-Carathéodory,

$$\beta(f(t, s, M)) \leq w(t, s, \beta(M))$$

for a.e. $s \in J$, $M \subset B_R$, and the unique $\psi \in C(J; [0, R])$ satisfying

$$\psi(t) \leq \int_0^T w(t, s, \psi(s))ds, \quad t \in J, \tag{1.8}$$

is $\psi \equiv 0$. Then the operator $N : \overline{U} \to C(J; Y)$ given by (1.5) is continuous and satisfies (1.4).

Theorem 1.6. ([8]) *Suppose that, for each $t \in J$, $f(t,\cdot,\cdot)$ is L^1-Carathéodory uniformly in t in the sense that there exists a bounded function $\eta : J^2 \to \mathbb{R}_+$ with $\eta(t,t') \to 0$ as $t - t' \to 0^+$ and*

$$\int_{t'}^{t} \sup_{|x|\le R} |f(t,s,x)|ds \le \eta(t,t')$$

for $0 \le t' < t \le T$. Also, suppose that, for each $t \in J$,

$$\int_{0}^{t_0} \sup_{|x|\le R} |f(t,s,x) - f(t',s,x)|ds \to 0 \quad \text{as } t' \to t,$$

where $t_0 = \min\{t, t'\}$ and that there exists $w : J^2 \times [0,R] \to \mathbb{R}$ such that, for each $t \in [0,T]$, $w(t, .\,, .)$ is L^1-Carathéodory,

$$\beta(f(t,s,M)) \le w(t,s,\beta(M))$$

for a.e. $s \in J$ and every $M \subset B_R$, and the unique solution $\psi \in C(J;\,[0,R])$ of the inequality

$$\psi(t) \le \int_0^t w(t,s,\psi(s))ds, \quad t \in J, \tag{1.9}$$

is $\psi \equiv 0$. Then the operator $N : \overline{U} \to C(J;Y)$ given by (1.6) *is continuous and satisfies* (1.4).

Now we shall refer to the operator (1.7). First, we introduce the following definition. A function $\chi : J \times D \to Y$, where $D \subset X$ and X, Y are two Banach spaces, is said to be *(q,p)-Carathéodory* ($1 \le q \le \infty$, $1 \le p < \infty$) if $\chi(\cdot,x)$ is strongly measurable for each $x \in D$, $\chi(t,\cdot)$ is continuous for a.e. $t \in J$ and

$$|\chi(t,x)|_Y \le \chi_0(t) + \alpha|x|_X^p$$

for a.e. $t \in J$ and all $x \in D$, where $\chi_0 \in L^q(J;\mathbb{R}_+)$ and $\alpha \in \mathbb{R}_+$.

Suppose that $f : J \times H \to H$ and consider the superposition operator associated to f, $F(y) = f(\cdot, y(\cdot))$.

Theorem 1.7. ([10]) *Assume that*

(i) *$k \in L^p(J \times J;\mathbb{R}_+)$ ($2 < p < \infty$) and the map $K : L^q(J;H) \to L^p(J;H)$ given by*

$$K(z)(t) = \int_0^T k(t,s)z(s)ds$$

($1/p + 1/q = 1$) *is well defined and splits into* $K = AA^*$ *with* $A : L^2(J; H) \to L^p(J; H)$ *and* A^* *the adjoint of* A,

(ii) f *is* $(q, p-1)$*-Carathéodory with*

$$|f(t,x)| \le f_0(t) + a|x|^{p-1}, \quad x \in H, \quad \text{a.e. } t \in J,$$

where $f_0 \in L^q(J; \mathbb{R}_+)$, $a \in \mathbb{R}_+$, *also there exists* $g : J \times H \to \mathbb{R}$ $(1,p)$*-Carathéodory such that* $g(t, 0) = 0$ *and*

$$g(t, x+y) - g(t, x) = (f(t, x), y) + \omega(t, x, y)$$

for a.e. $t \in J$ *and all* $x, y \in H$, *where*

$$\omega(t, x, y)/|y| \to 0 \quad \text{as } y \to 0$$

uniformly for $x \in H$ *and a.e.* $t \in J$,

(iii) *there exists* $R > 0$ *and* $\epsilon > 0$ *such that, for each* $\eta \in L^p(J; \mathbb{R}_+)$ *satisfying*

$$|\eta|_p \le \epsilon + C|k|_{L^p(J^2)},$$

where

$$C = |f_0|_q + a|A|^{p/q} R^{p/q}, \tag{1.10}$$

there exists a $(q, p-1)$*-Carathéodory function* $w_\eta : J \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\beta(f(t, M)) \le w_\eta(t, \beta(M)) \tag{1.11}$$

for every set $M \subset H$ *with* $|M| \le \eta(t)$ *a.e.* $t \in J$, *and* $\psi \equiv 0$ *is the unique solution in* $L^p(J; \mathbb{R}_+)$ *of the inequality*

$$\psi(t) \le \int_0^T k(t,s) w_\eta(s, \psi(s)) ds, \quad \text{a.e. } t \in J, \tag{1.12}$$

(*here* $|M| = \sup\{|x| : x \in M\}$).

Then the functional

$$E : L^2(J; H) \to \mathbb{R}, \quad E(x) = |x|_2^2/2 - GA(x), \tag{1.13}$$

where

$$G : L^p(J; H) \to \mathbb{R}, \quad G(y) = \int_0^T g(s, y(s)) ds,$$

belongs to $C^1(L^2(J;H);\mathbb{R})$, $E' = I - A^*FA$, *and satisfies the* $(S\text{-}P\text{-}S)_R$*-condition.*

Notice that, in [10], we asked that g be (∞, p)-Carathéodory. However, the proof shows that the result remains valid if g is more generally $(1,p)$-Carathéodory.

If we examine the above three theorems, we see that a common assumption is that a certain integral inequality (1.8), (1.9), respectively, (1.12), has no other nonnegative solution than the null function. So it is important to find sufficient conditions for that this happens. We shall obtain such conditions in the next section.

2. Integral Inequalities without Non-zero Solutions

In this section we are concerned with the following implications:

$$\left[\begin{matrix}\psi(t) \le \int_0^T w(t,s,\psi(s))ds, \ t \in J = [0,T] \\ \psi \in \Psi\end{matrix} \Longrightarrow \psi \equiv 0\right] \tag{2.1}$$

and, in particular, for $w(t,s,r) = 0$ when $t < s$, with the implication

$$\left[\begin{matrix}\psi(t) \le \int_0^t w(t,s,\psi(s))ds, \ t \in J \\ \psi \in \Psi\end{matrix} \Longrightarrow \psi \equiv 0\right]. \tag{2.2}$$

Here Ψ is a class of nonnegative real functions on J.

First we collect sufficient conditions for (2.1):

Let us consider $R \in (0,\infty]$, $p \in [1,\infty]$, $q \in [1,\infty]$ with $1/p + 1/q = 1$, and $w : J^2 \times (0,R] \to (0,\infty)$. We adopt the convention that if $R = \infty$, then by $(0,R]$, $[0,R]$ we mean the intervals $(0,\infty)$ and $[0,\infty)$, respectively. Also, throughout this section, by $|.|_p$ we shall mean the L^p- norm.

Theorem 2.1. (1) *Assume that one of the following conditions is satisfied:*

(i) $w(t,\cdot,r) \in L^1(J)$, $w(t,s,\cdot)$ *is continuous, nondecreasing for any* $r \in (0,R]$, $t,s \in J$, *and*

$$1 < r/|w(t,\cdot,r)|_1 \tag{2.3}$$

for all $r \in (0,R]$ *and* $t \in J$.

(ii) $w(t,s,r) = k(t,s)w_1(s,r)$, $k : J^2 \to \mathbb{R}_+$, $k(t,\cdot) \in L^q(J)$ *for every* $t \in J$, $w_1 : J \times (0,R] \to (0,\infty)$, $w_1(\cdot,r) \in L^p(J)$, $w_1(s,\cdot)$ *is continuous, nondecreasing for every* $r \in (0,R]$, $s \in J$, *and*

$$|k(t,\cdot)|_q < r/|w_1(\cdot,r)|_p$$

for all $r \in (0,R]$ *and* $t \in J$.

(iii) $w(t,s,r) = k(t,s)w_0(r)$, $k : J^2 \to \mathbb{R}_+$, $k(t,\cdot) \in L^1(J)$ *for every* $t \in J$, $w_0 : (0,R] \to (0,\infty)$ *is continuous, nondecreasing, and*

$$|k(t,\cdot)|_1 < r/w_0(r) \tag{2.4}$$

for all $r \in (0,R]$, $t \in J$.

Then (2.1) holds for $\Psi = C(J;[0,R])$.

(2) *Assume that one of the following conditions is satisfied:*

(iv) $w(t,s,r) = k(t,s)r$, $k : J^2 \to \mathbb{R}_+$, $k(t,\cdot) \in L^q(J)$ *for every* $t \in J$, *the map* $t \mapsto |k(t,\cdot)|_q$ *belongs to* $L^p(J)$ *and*

$$\big||k(t,\cdot)|_q\big|_p < 1. \tag{2.5}$$

(v) $w(t,s,r) = k(t,s)\delta(s)r$, *where* k *is the Green function of the problem* $-u'' = f$, $u(0) = u(T) = 0$, $\delta \in L^\tau(J;\mathbb{R}_+)$, *and*

$$\frac{1}{m-1}|\delta|_\tau < \inf\left\{ \frac{\int_0^T |u|^{m-2}u'^2 dt}{\left(\int_0^T |u|^{m\tau/(\tau-1)}dt\right)^{(\tau-1)/\tau}} : u \in C_0^1(J),\ u \neq 0 \right\}, \tag{2.6}$$

where $p \in (1,\infty)$, $\tau \in [q,\infty)$ *and* $m \in [2,\infty)$.

Then (2.1) *holds for* $\Psi = L^p(J;\mathbb{R}_+)$.

Proof. Assume (i). Suppose that there is $\psi \in C(J;[0,R])$, $\psi \neq 0$, with

$$\psi(t) \leq \int_0^T w(t,s,\psi(s))\,ds, \quad t \in J. \tag{2.7}$$

Let $r_0 = \psi(t_0) = \max_{t\in J}\psi(t)$. Since $\psi \neq 0$, we have $r_0 \in (0,R]$. Then, since $w(t_0,s,\cdot)$ is nondecreasing, we have

$$r_0 \leq \int_0^T w(t_0,s,\psi(s))ds \leq \int_0^T w(t_0,s,r_0)ds = |w(t_0,\cdot,r_0)|_1.$$

Hence $r_0/|w(t_0,\cdot,r_0)|_1 \leq 1$, which contradicts (2.3).

It is easily seen that (iii) implies (ii) for $q = 1$, and (ii) implies (i) via Hölder's inequality.

Assume (iv) and suppose that (2.7) holds for some non-zero $\psi \in L^p(J;\mathbb{R}_+)$. Then, using Hölder's inequality, we see that

$$\psi(t) \leq |k(t,\cdot)|_q|\psi|_p, \quad t \in J.$$

It follows that

$$|\psi|_p \le |\psi|_p \| k(t,\cdot)|_q|_p.$$

Since $\psi \neq 0$, $|\psi|_p \neq 0$. Consequently, we have

$$1 \le \| k(t,\cdot)|_q|_p,$$

which contradicts (2.5).

Finally, assume that (v). Suppose (2.7) holds for some non-zero $\psi \in L^p(J;\mathbb{R}_+)$. Let

$$u(t) = \int_0^T k(t,s)\delta(s)\psi(s)ds.$$

Then $u \in C_0^1(J;\mathbb{R}_+)$, $\psi \le u$ on J, $u \neq 0$, and

$$-u''(t) = \delta(t)\psi(t), \quad \text{a.e. } t \in J.$$

It follows that

$$-u''(t) \le \delta(t)u(t), \quad \text{a.e. } t \in J.$$

If we multiply by $u(t)^{m-1}$ and integrate, we obtain

$$(m-1)\int_0^T u^{m-2}u'^2 dt \le \int_0^T \delta u^m dt \le |\delta|_\tau (\int_0^T u^{m\tau/(\tau-1)}dt)^{(\tau-1)/\tau}.$$

It is clear that this inequality is in contradiction with (2.6). This completes the proof.

Notice that (2.6) is possible since the infimum in the right hand side is positive as shows the following inequality of Wirtinger type:

Theorem 2.2. *Assume $\tau \in (1,\infty)$ and $m \in [2,\infty)$. For each $u \in C_0^1[0,T]$, one has*

$$\left(\int_0^T |u|^{m\tau/(\tau-1)}\,dt\right)^{(\tau-1)/\tau} \le c\int_0^T |u|^{m-2}u'^2\,dt, \tag{2.8}$$

where

$$c = c(\tau,m,T) = \left(\frac{m}{2}\right)^2 \left(\frac{\tau-1}{2\tau-1}\right)^{(\tau-1)/\tau} T^{(2\tau-1)/\tau}. \tag{2.9}$$

Proof. We have

$$\frac{2}{m}|u(t)|^{m/2} = \int_0^t |u|^{(m-4)/2} u\, u'\, ds$$
$$\leq \left(\int_0^t 1^2\, ds\right)^{1/2} \left(\int_0^t |u|^{m-2} u'^2\, ds\right)^{1/2}$$
$$\leq t^{1/2}\left(\int_0^T |u|^{m-2} u'^2\, ds\right)^{1/2}.$$

If we take the $2\tau/(\tau-1)$-th power and integrate, we obtain

$$\left(\frac{2}{m}\right)^{2\tau/(\tau-1)} \int_0^T |u|^{m\tau/(\tau-1)} dt$$
$$\leq \frac{\tau-1}{2\tau-1} T^{(2\tau-1)/(\tau-1)} \left(\int_0^T |u|^{m-2} u'^2 dt\right)^{\tau/(\tau-1)}.$$

This inequality is equivalent to (2.8) with c given by (2.9). This completes the proof.

An interesting open problem is to find the smallest value c^* of c in the inequality (2.8). Note that, for $m = \tau = 2$, (2.8) was first established in Aramă [1] with the better constant $c = T^{3/2}\sqrt{2}/\pi$. Also, for $m = 2$ and any $\tau > 1$, (2.8) can be derived from a more general inequality of Boyd [2] (see also [5], p. 149–150).

It is clear that, if $w(t,s,r) = 0$ whenever $t < s$, then (2.1) turns into (2.2). Thus, in particular, each condition from (i) to (iv) is also sufficient for (2.2). However, (iii) and (iv) can be relaxed for (2.2) as shows the next result.

Theorem 2.3. (1) *Assume* $p < \infty$ *and*
(iii*) $w(t,s,r) = k(t,s)w_0(r)$, $k : J^2 \to \mathbb{R}_+$, $k(t,\cdot) \in L^q(J)$ *for each* $t \in J$, $\sup_{t\in J} |k(t,\cdot)|_q < \infty$, $w_0 : (0,R] \to (0,\infty)$ *is continuous, nondecreasing, and*

$$\int_{0+} \frac{r^{p-1}}{w_0(r)^p}\, dr = \infty. \tag{2.10}$$

Then (2.2) *holds for* $\Psi = C(J;[0,R])$.
(2) *Assume that*
(iv*) $w(t,s,r) = k(t,s)r$, *with* k *as in* (iii*).
Then (2.2) *holds for* $\Psi = L^p(J;\mathbb{R}_+)$.

Proof. (1) Assume (iii*). Let $\alpha = \sup_{t\in J} |k(t,\cdot)|_q$. Suppose that $\psi \in C(J;[0,R])$, $\psi \neq 0$, and

$$\psi(t) \leq \int_0^t k(t,s)w_0(\psi(s))ds, \quad t \in J.$$

We have

$$\psi(t) \le \left(\int_0^t k(t,s)^q ds\right)^{1/q}\left(\int_0^t w_0(\psi(s))^p ds\right)^{1/p}$$
$$\le c(t) := \alpha\left(\int_0^t w_0(\psi(s))^p ds\right)^{1/p}.$$

Then

$$(c(t)^p)' = p\, c(t)^{p-1}c'(t) = \alpha^p w_0(\psi(t))^p \le \alpha^p w_0(c(t))^p. \tag{2.11}$$

It is clear that c is a continuous and nondecreasing function. Consequently, since $\psi \not\equiv 0$, $c(T) > 0$. In addition, $c(0) = 0$. Then, for each $\varepsilon \in (0, A)$, where $A = \min\{c(T),\ R\}$, there is a subinterval $[a, b] \subset J$ with

$$c(a) = \varepsilon,\ c(b) = A, \quad c(t) \in (\varepsilon, A),\ t \in (a, b).$$

Then, from (2.11), we deduce

$$p\int_\varepsilon^A \frac{r^{p-1}}{w_0(r)^p}dr = \int_a^b \frac{(c(s)^p)'}{w_0(c(s))^p}ds \le \alpha^p T.$$

Letting $\varepsilon \to 0^+$, we derive a contradiction to (2.10).

(2) Use Gronwall's inequality. This completes the proof.

Notice in the case $\gamma := \sup_{t\in J} |k(t,\cdot)|_1 > 0$, the condition (2.10) is less restrictive than (2.4). Indeed, if (2.4) holds, then $\inf_{r\in(0,R]}(r/w_0(r)) \ge \gamma > 0$ and, in consequence,

$$\int_{0+}(r^{p-1}/w_0(r)^p)dr \ge \inf_{r\in(0,R]}(r/w_0(r))^p \int_{0+} r^{-1}dr = \infty.$$

3. Application

Consider the superlinear boundary value problem

$$\begin{cases} -y'' = |B(y)|^{p-2}B^*B(y) + h(t), & \text{a.e. } t \in J, \\ y(0) = y(T) = 0 \end{cases} \tag{3.1}$$

in a real Hilbert space H, where $B : H \to H$ is a linear isomorphism, B^* is the adjoint of B, $p > 2$ and $h \in L^q(J; H)$. By a solution of (3.1), we mean a function $y \in C_0^1(J; H)$ such that y' is absolutely continuous and y satisfies (3.1).

With the notations of Theorem 1.7, the problem (3.1) is equivalent to the Hammerstein integral equation $y = KF(y)$, where the kernel k of the linear mapping K is the Green function associated to the operator $-y''$ and to the boundary conditions $y(0) = y(T) = 0$. Also $f(t,x) = |B(x)|^{p-2}B^*B(x) + h(t)$. We can see that the conditions (i)-(ii) of Theorem 1.7 are fulfilled with $f_0 = |h|$, $a = |B|^p$, $g(t,x) = p^{-1}|B(x)|^p + (h(t),x)$ and A the square root of K. It is clear that, if $x \in L^2(J;H)$ is a solution of the equation $x - A^*FA(x) = 0$, then $y = A(x)$ is a solution of (3.1). Also note $|A| = |A^*|$. In what follows we show that the condition (iii) of Theorem 1.7 also holds.

First, we observe that, if $M \subset H$ and $|M| \le \eta(t)$, then

$$\beta(f(t,M)) \le (p-1)|B|^p\eta(t)^{p-2}\beta(M). \tag{3.2}$$

Indeed, for every x, $x' \in M$, we have

$$\begin{aligned}
&|f(t,x) - f(t,x')| \\
&= \big||B(x)|^{p-2}B^*B(x) - |B(x')|^{p-2}B^*B(x')\big| \\
&\le \big||B(x)|^{p-2}B^*B(x-x')\big| + \big|(|B(x)|^{p-2} - |B(x')|^{p-2})B^*B(x')\big| \\
&\le |B|^p|M|^{p-2}|x-x'| + |B|^2|M|\big||B(x)|^{p-2} - |B(x')|^{p-2}\big| \\
&\le |B|^p|M|^{p-2}|x-x'| + (p-2)|B|^p|M|^{p-2}|x-x'| \\
&= (p-1)|B|^p|M|^{p-2}|x-x'|.
\end{aligned}$$

It follows that $\beta(f(t,M)) \le (p-1)|B|^p|M|^{p-2}\beta(M)$ and so (3.2) holds.

Thus the function w_η in (1.11) is

$$w_\eta(t,s) = (p-1)|B|^p\eta(t)^{p-2}s.$$

Note that the constant C in (1.10) is $|h|_q + |B|^p|A|^{p-1}R^{p-1}$ (recall that $p/q = p-1$). We wish to show that there exist $\epsilon > 0$ and $R > 0$ such that the null function be the unique solution in $L^p(J;\mathbb{R}_+)$ of the inequality

$$\psi(t) \le (p-1)|B|^p \int_J k(t,s)\eta(s)^{p-2}\psi(s)ds, \quad t \in J,$$

where $\eta \in L^p(J;\mathbb{R}_+)$ and

$$|\eta|_p \le \epsilon + (|h|_q + |B|^p|A|^{p-1}R^{p-1})|k|_{L^p(J^2)}. \tag{3.3}$$

Since $\eta \in L^p(J)$, we have $\eta^{p-2} \in L^{p/(p-2)}(J)$. Thus we may apply Theorem 2.1 (v), if we take $\delta(s) = (p-1)|B|^p\eta(s)^{p-2}$ and $\tau = p/(p-2)$. We shall consider

$m = p$, although it is clear that an analogous result can be established for any $m \geq 2$. Then (2.6) means

$$|B|^p|\eta|_p^{p-2} < \lambda_{p-1} := \inf\left\{\frac{\int_0^T |u|^{p-2}u'^2dt}{(\int_0^T |u|^{p^2/2}dt)^{2/p}} : u \in C_0^1(J), u \neq 0\right\}$$

or, equivalently,

$$|\eta|_p < |B|^{-p/(p-2)}\lambda_{p-1}^{1/(p-2)}.$$

Clearly, according to (3.3), the last inequality holds if

$$(|h|_q + |B|^p|A|^{p-1}R^{p-1})|k|_{L^p(J^2)} < |B|^{-p/(p-2)}\lambda_{p-1}^{1/(p-2)}. \tag{3.4}$$

For $|h|_q$ sufficiently small that

$$|h|_q < |k|_{L^p(J^2)}^{-1}|B|^{-p/(p-2)}\lambda_{p-1}^{1/(p-2)}, \tag{3.5}$$

the inequality (3.4) is equivalent to

$$\begin{aligned} R &< R_0 \\ &:= |B|^{-p/(p-1)}|A|^{-1}\left(|k|_{L^p(J^2)}^{-1}|B|^{-p/(p-2)}\lambda_{p-1}^{1/(p-2)} - |h|_q\right)^{1/(p-1)}. \end{aligned} \tag{3.6}$$

Thus we have proved the following result:

Theorem 3.1. *Assume* (3.5). *Then the energy functional* $E : L^2(J;H) \to \mathbb{R}$ *of the problem* (3.1) *given by* (1.13) *satisfies the* (S-P-S)$_R$*-condition for any* R *satisfying* (3.6).

Using the above theorem, we can obtain an alternative result for (3.1):

Theorem 3.2. *Assume* (3.5). *Then at least one of the following two statements is valid.*

(i) *For each* $R \in (0, R_0)$, *there exists* $\mu \in (0,1)$ *and* $z \in L^2(J;H)$ *such that* $|z|_2 = \mu^{1/(p-1)}R$ *and* $y = A(z)$ *solves*

$$\begin{cases} -y'' = |B(y)|^{p-2}B^*B(y) + \mu h(t), & \text{a.e. } t \in J, \\ y(0) = y(T) = 0. \end{cases} \tag{3.7}$$

(ii) *There exists* $x \in L^2(J;H)$ *such that* $|x|_2 < R_0$ *and* $y = A(x)$ *solves* (3.1).

Proof. Suppose that (i) does not hold. Then there exists $R \in (0, R_0)$ such that for every $\mu \in (0,1)$ and $z \in L^2(J;H)$ with $|z|_2 = \mu^{1/(p-1)}R$, the function $y = A(z)$ does not satisfy (3.7), that is,

$$y \neq K(FA(z) + (\mu - 1)h)$$

or, equivalently,

$$z \neq A^*(FA(z) + (\mu - 1)h). \tag{3.8}$$

Now, if, for any $x \in L^2(J;H)$ with $|x|_2 = R$ and $\lambda \in (0,1)$, we put $\mu = \lambda^{(p-1)/(p-2)}$ and $z = \lambda^{1/(p-2)}x$, then (3.8) guarantees

$$x \neq \lambda(x - E'(x)).$$

Thus the Leray-Schauder boundary condition required by Theorem 1.4 is satisfied. Now, (ii) follows from Theorem 1.4 (b) if we use Theorem 3.1 and we observe that $\inf_{|x|_2 \le R} E(x) > -\infty$ and that $-(E'(x), x) \le C$ for $|x|_2 = R$.

Finally, we shall discuss the case $h = 0$, when $y = 0$ is a solution of (3.1). We shall try to prove the existence of a non-zero solution via Theorem 1.4 (a). For this, take any $R \in (0, R_0)$ so that the $(S-P-S)_R$-condition is satisfied. Two cases are possible:

(1) The Leray-Schauder boundary condition does not hold. Then there exists an $x \in L^2(J;H)$ with $|x|_2 = R$ and a $\lambda \in (0,1)$ such that $x = \lambda(x - E'(x))$, i.e., $x = \lambda A^*FA(x)$. Recall $F(y) = |B(y)|^{p-2}B^*B(y)$. It is easily seen that $\overline{x} = \lambda^{1/(p-2)}x$ satisfies $\overline{x} = A^*FA(\overline{x})$, i.e., $y = A(\overline{x})$ is a non-zero solution of (3.1).

(2) The Leray-Schauder boundary condition holds. Then we look for a function x_1 and a number $r \in (0,R)$ such that $r < |x_1|_2 < R$ and $E(x_1) \le \inf_{|x|_2 = r} E(x)$. Let $\varepsilon > 0$ and $x_2 \in L^2(J;H)$ with $|x_2|_2 = 1$ and $|BA(x_2)|_p^p \ge |BA|^p - \varepsilon$. We look for x_1 in the form $x_1 = \lambda x_2$, where $\lambda \in (r, R)$. Let us consider the function

$$\phi(\sigma) = \sigma^2/2 - p^{-1}\sigma^p|BA|^p, \quad \sigma \ge 0.$$

We have

$$\begin{aligned} E(x_1) &= \lambda^2/2 - p^{-1}\lambda^p|BA(x_2)|_p^p \\ &\le \lambda^2/2 - p^{-1}\lambda^p|BA|^p + p^{-1}\lambda^p\varepsilon \\ &= \phi(\lambda) + p^{-1}\lambda^p\varepsilon. \end{aligned}$$

Also, for every $x \in L^2(J;H)$ with $|x|_2 = r$, one has

$$E(x) = r^2/2 - p^{-1}r^p|BA(r^{-1}x)|_p^p \ge r^2/2 - p^{-1}r^p|BA|^p = \phi(r).$$

It is easy to check that the function ϕ is increasing on $[0, \sigma_M]$ and decreasing on $[\sigma_M, \infty)$, where $\sigma_M = |BA|^{-p/(p-2)}$. Consequently, if $\sigma_M < R$, we may choose

$r = \sigma_M$, λ any number in (r, R) and $\varepsilon > 0$ sufficiently small, such that $\phi(\lambda) + p^{-1}\lambda^p\varepsilon \leq \phi(r)$. Therefore, according to Theorem 1.4 (a), a sufficient condition for the existence of non-zero solutions to (3.1) (when $h = 0$) is that $\sigma_M < R_0$, or, equivalently,

$$1 < \lambda_{p-1}|BA|^{p(p-1)}|A|^{-(p-1)(p-2)}|B|^{-p(p-1)}|k|^{2-p}_{L^p(J^2)}. \tag{3.9}$$

An open problem is to study if (3.9) is possible or not, and also to refine the reasoning above in order to obtain inequalities less restrictive than (3.9).

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