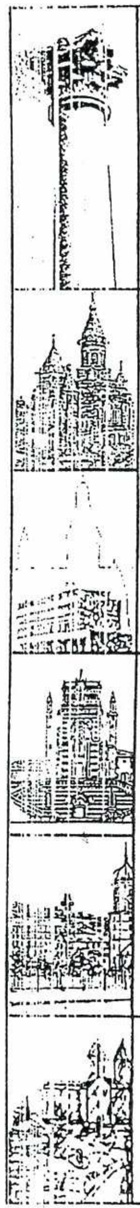

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THE CONTINUATION PRINCIPLE FOR GENERALIZED CONTRACTIONS

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The Continuation Principle for Generalized Contractions

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ABSTRACT: A continuation principle for contractions on spaces endowed with vector-valued metrics is presented together with an application to Hammerstein integral equations in $\mathbb{R}^n$ with matrix-valued kernels.

KEY WORDS: Contraction, Generalized metric space, Continuation, Fixed point, Hammerstein integral equation.
AMS SUBJECT CLASSIFICATION CODE: 45G15, 47H10, 54H25.

1 The continuation principle

The Banach contraction principle was generalized by Perov (see Perov-Kidenko [3] and Rus [6]) for contractive maps on spaces endowed with vector-valued metrics. Also, Granas [1] proved that the property of having a fixed point is invariant by homotopy for contractions on complete metric spaces. This result was completed in Precup [4] (see also O'Regan-Precup [2]) by an iterative procedure of discrete continuation along the fixed points curve. The result was recently extended to contractions on spaces endowed with vector-valued metrics in Precup [5]. In this paper the main result from [5] is presented together with an application to Hammerstein integral equations in $\mathbb{R}^n$ with matrix-valued kernels.

Let X be a nonempty set. By a *vector-valued metric* on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, by $x \leq y$ we mean that $x_i \leq y_i$ for $i = 1, 2, ..., n$.

A set X endowed with a vector-valued metric d is said to be a *generalized metric space*. For the generalized metric spaces, the notions of a convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 1 *Let (X, d) be a generalized metric space. A map $T : X \to X$ is said to be contractive if there exists a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ such that*

$$M^j \to 0 \text{ as } j \to \infty \quad (1)$$

and

$$d(T(u), T(v)) \leq M\, d(u, v)$$

for all $u, v \in X$. A matrix M which satisfies (1) is said to be convergent to zero.

It is known (see Rus [6]) that a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ is convergent to zero if and only if $I - M$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots.$$

Theorem 2 (Perov) *Let (X, d) be a complete generalized metric space and $T : X \to X$ be contractive with the Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ one has*

$$d(T^j(u_0), u^*) \leq M^j (I - M)^{-1} d(u_0, T(u_0))$$

for every $j \in \mathbb{N}$.

We now state the continuation principle for such type of mappings which was established in Precup [5].

Theorem 3 *Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}^n$ and U be an open set of X. Let $H : \overline{U} \times [0, 1] \to X$ and assume that the following conditions are satisfied:*

(a1) there is a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ convergent to zero such that

$$d(H(u, \lambda), H(v, \lambda)) \leq M\, d(u, v)$$

for all $u, v \in \overline{U}$ and $\lambda \in [0, 1]$;

(a2) $H(u, \lambda) \neq u$ for all $u \in \partial U$ and $\lambda \in [0, 1]$;

(a3) H is continuous in λ, uniformly for $u \in \overline{U}$, i.e. for each $\varepsilon \in (0, \infty)^n$ and $\lambda \in [0, 1]$, there is $\rho \in (0, \infty)$ such that $d(H(u, \lambda), H(u, \mu)) < \varepsilon$ whenever $u \in \overline{U}$ and $|\lambda - \mu| < \rho$.

In addition suppose that $H_0 := H(., 0)$ has a fixed point. Then, for each $\lambda \in [0, 1]$, there exists a unique fixed point $u(\lambda)$ of $H_\lambda := H(., \lambda)$. Moreover, $u(\lambda)$ depends continuously on λ and there exists $r \in (0, \infty)^n$, integers $m, k_1, k_2, ..., k_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ such that for any $u_0 \in X$ satisfying $d(u_0, u(0)) \leq r$, the sequences $(u_{j,i})_{i\geq 0}$, $j = 1, 2, ..., m$,

$$u_{1,0} = u_0$$
$$u_{j,i+1} = H_{\lambda_j}(u_{j,i}), \quad i = 0, 1, \ldots$$
$$u_{j+1,0} = u_{j,k_j}, \quad j = 1, 2, ..., m - 1$$

are well defined and satisfy

$$d(u_{j,i}, u(\lambda_j)) \leq M^i (I - M)^{-1} d(u_{j,0}, H_{\lambda_j}(u_{j,0})), \quad i \in \mathbb{N}.$$

The proof of Theorem 3 also yields the following algorithm for the approximation of $u(1)$:

Suppose we know $r \in (0, \infty)^n$ and the number $h > 0$ such that $(I - M) r \in (0, \infty)^n$ and

$$d(u, H(u, \lambda)) \leq (I - M) r$$

whenever $u = H(u, \mu)$ and $|\lambda - \mu| \leq h$. We wish to obtain an approximation $\overline{u}_1$ of $u(1)$ with $d(\overline{u}_1, u(1)) \leq \varepsilon$ for some $\varepsilon \in (0, \infty)^n$. Then we choose any partition $0 = \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ of $[0, 1]$ with $\lambda_{j+1} - \lambda_j \leq h$, $j = 0, 1, ..., m - 1$, any element u_0 with $d(u_0, u(0)) \leq r$ and we follow the next

Iterative procedure:

Set $k_0 := 0$ and $u_{0,k_0} := u_0$;
For $j := 1$ to $m - 1$ do
  $u_{j,0} := u_{j-1,k_{j-1}}$
  $i := 0$
  While $M^i (I - M)^{-1} d(u_{j,0}, H_{\lambda_j}(u_{j,0})) \nleq r$
    $u_{j,i+1} := H_{\lambda_j}(u_{j,i})$
    $i := i + 1$
  $k_j = i$

Set $i := 0$
While $M^i (I - M)^{-1} d(u_{m,0}, H_1(u_{m,0})) \not\leq \varepsilon$
$u_{m,i+1} = H_1(u_{m,i})$
$i := i + 1$
Finally take $\overline{u}_1 = u_{m,i}$.

Notice for $m = 1$, Theorem 3 reduces to Corollary 2.5 in Precup [4].

2 Hammerstein Integral Equations with Matrix-valued Kernels

In this section we give an application of Theorem 3 to the Hammerstein integral equation in $\mathbf{R}^n$

$$u(x) = \int_\Omega \kappa(x, y) f(y, u(y))\, dy, \quad x \in \Omega, \tag{2}$$

in the case that the kernel κ has matrix-values, i.e.

$$\kappa : \Omega^2 \to M_{n\times n}(\mathbf{R}), \; \kappa = [\kappa_{ij}].$$

The usual Hammerstein equation in $\mathbf{R}^n$ with a scalar kernel appears as a particular case of (2) when $\kappa_{ij} = 0$ for $i \neq j$ and $\kappa_{ii} = \kappa_{jj}$ for all $i, j \in \{1, 2, ..., n\}$.

The simplest examples of problems which allow us to systems of the form (2) with matrix-valued kernels are the boundary value problems for differential equations of order ≥ 2. For instance, the problem

$$\begin{cases} u'' = g(x, u, u'), \; x \in [0, 1] \\ u(0) = 0, \; u'(1) = 0 \end{cases}$$

can be put in the form (2) if we let $n = 2$, $u_1 = u$, $u_2 = u'$,

$$\kappa_{11}(x, y) = \begin{cases} 1, \; y \leq x \\ 0, \; y > x \end{cases}, \; \kappa_{22}(x, y) = \begin{cases} 0, \; y \leq x \\ -1, \; y > x \end{cases}$$
$$\kappa_{12} = \kappa_{21} = 0,$$

and $f_1(x, u_1, u_2) = u_2$, $f_2(x, u_1, u_2) = g(x, u_1, u_2)$.

Before we state the main result we introduce the following notations. For an element $z \in \mathbf{R}^n$ we let

$$\|z\| = (|z_1|, |z_2|, ..., |z_n|).$$

Also, for a function $u \in L^p(\Omega; \mathbf{R}^n)$ $(1 \leq p \leq \infty)$ we let

$$\|u\|_p = \left(|u_1|_p, |u_2|_p, ..., |u_n|_p\right).$$

Clearly $\|.\|$ and $\|.\|_p$ are vector-valued norms on $\mathbf{R}^n$ and $L^p(\Omega; \mathbf{R}^n)$, respectively. Endowed with the vector-valued metric $d_p(u, v) = \|u - v\|_p$, $L^p(\Omega; \mathbf{R}^n)$ is a complete generalized metric space. Similarly, $\left(C(\overline{\Omega}; \mathbf{R}^n), d_\infty\right)$ is a complete generalized metric space.

We now state and prove a general existence and uniqueness principle for (2).

Theorem 4 *Let $\Omega \subset \mathbf{R}^N$ be an open, bounded set, $\kappa : \Omega^2 \to M_{n\times n}(\mathbf{R})$ measurable and $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$. Suppose that there are $p \in [2, \infty]$, $q \in [1, \infty)$, $p \geq q$, and an open subset U of $\left(L^p(\Omega; \mathbf{R}^n), \|.\|_p\right)$ containing the origin, such that the following conditions are satisfied:*

$$\begin{cases} (a) \text{ if } 1 \leq p < \infty, \text{ then } \kappa_{ij}(x, .) \in L^r(\Omega) \text{ a.e. } x \in \Omega \text{ and} \\ \text{the map } x \longmapsto |\kappa_{ij}(x, .)|_r \text{ belongs to } L^p(\Omega) \; (1/q + 1/r = 1); \\ (b) \text{ if } p = \infty, \text{ then } \kappa_{ij}(x, .) \in L^r(\Omega) \text{ for every } x \in \Omega \text{ and} \\ \text{the map } x \longmapsto \kappa_{ij}(x, .) \text{ is continuous from } \overline{\Omega} \text{ to } L^r(\Omega); \end{cases} \tag{3}$$

$$\begin{cases} f \text{ satisfies the Carathéodory conditions, } f(., 0) \in L^q(\Omega; \mathbf{R}^n) \text{ and} \\ \|f(y, z_1) - f(y, z_2)\| \leq L(y) \|z_1 - z_2\| \\ \text{a.e. } y \in \Omega, \text{ for all } z_1, z_2 \in \mathbf{R}^n \text{ and some } L \in L^{pq/(p-q)}(\Omega; M_{n\times n}(\mathbf{R}_+)). \end{cases} \tag{4}$$

Let $M = [M_{ik}]$,

$$M_{ik} = \sum_{j=1}^{n} \left\| |\kappa_{ij}(x, .)|_r \right|_p |L_{jk}|_{pq/(p-q)}$$

and assume that M is convergent to zero. In addition suppose

$$u \in U$$

for any solution $u \in \overline{U}$ to

$$u(x) = \lambda \int_\Omega \kappa(x, y) f(y, u(y))\, dy, \quad \text{a.e. } x \in \Omega$$

for each $\lambda \in (0, 1)$. Then (2) has a unique solution $u \in \overline{U} \subset L^p(\Omega; \mathbf{R}^n)$. Moreover, for $p = \infty$, $u \in C(\overline{\Omega}; \mathbf{R}^n)$.

Proof. Apply Theorem 3 to $X = L^p(\Omega; \mathbf{R}^n)$ with norm $\|.\|_p$ and $H : \overline{U} \times [0,1] \to L^p(\Omega; \mathbf{R}^n)$ given by

$$H(u,\lambda)(x) = \lambda \int_\Omega \kappa(x,y) f(y, u(y))\, dy \quad (x \in \Omega).$$

From (4) we have

$$\|f(y,z)\| \le \|f(y,0)\| + L(y)\|z\|.$$

Hence

$$|f_i(y,z)| \le |f_i(y,0)| + \sum_{j=1}^n L(y)_{ij}|z_j|. \tag{5}$$

By Young's Inequality,

$$L(y)_{ij}|z_j| \le \frac{L(y)_{ij}^{p/(p-q)}}{p/(p-q)} + \frac{|z_j|^{p/q}}{p/q}.$$

Since $f_i(.,0)$, $L(.)_{ij}^{p/(p-q)} \in L^q(\Omega)$, from (5) we get that

$$|f(y,z)| \le g(y) + c|z|^{p/q}$$

for some $g \in L^q(\Omega)$ and $c \ge 0$. Hence the Nemytskii operator associated to f maps $L^p(\Omega; \mathbf{R}^n)$ into $L^q(\Omega; \mathbf{R}^n)$. From (3) we see that the Fredholm linear integral operators of kernels κ_{ij} maps $L^q(\Omega; \mathbf{R}^n)$ into $L^p(\Omega; \mathbf{R}^n)$. Hence H is well-defined. Furthermore,

$$|(H_i(u,\lambda) - H_i(v,\lambda))(x)| \le \int_\Omega \sum_{j=1}^n |\kappa_{ij}(x,y)|\, |f_j(y,u(y)) - f_j(y,v(y))|\, dy$$

$$\le \int_\Omega \sum_{j=1}^n |\kappa_{ij}(x,y)| \sum_{k=1}^n L_{jk}(y)\, |u_k(y) - v_k(y)|\, dy$$

$$\le \sum_{k=1}^n \sum_{j=1}^n |\kappa_{ij}(x,.)|_r\, |L_{jk}|_{pq/(p-q)}\, |u_k - v_k|_p.$$

Consequently

$$|H_i(u,\lambda) - H_i(v,\lambda)|_p \le \sum_{k=1}^n \sum_{j=1}^n \left||\kappa_{ij}(x,.)|_r\right|_p |L_{jk}|_{pq/(p-q)}\, |u_k - v_k|_p$$

$$= \sum_{k=1}^n M_{ik}\, |u_k - v_k|_p.$$

Thus

$$\|H(u,\lambda) - H(v,\lambda)\|_p \le M\, \|u - v\|_p.$$

Now the conclusion follows from Theorem 3. ■

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