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FIXED POINT THEOREMS FOR ACYCLIC MULTIVALUED MAPS AND INCLUSIONS OF HAMMERSTEIN TYPE

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Abstract. The aim of this lecture is to present a new compactness method for operator inclusions in general, and for Hammerstein like inclusions, in particular. This method applies to acyclic multivalued maps which satisfy a generalized compactness condition of Mönch type.

Keywords: Multivalued map, acyclic map, Hammerstein operator, operator inclusion, compactness, fixed point.

1. The operator form of the initial and boundary value problems

STEP I: Consider the initial value problem (IVP) and the boundary value problem (BVP):

$$\text{(1)} \qquad \begin{cases} u' = f(t,u), & t \in I = [0,T] \\ u(0) = 0; \end{cases} \qquad \begin{cases} u'' = f(t,u), & t \in I \\ u \in \mathcal{B} \end{cases}$$

for a system of n differential equations. Here $\mathcal{B}$ stands for the boundary conditions. Under standard conditions, both problems (1) can be put under the operator form

$$u = N(u), \quad u \in C(I;\mathbf{R}^n),$$

where $N : C(I;\mathbf{R}^n) \to C(I;\mathbf{R}^n)$ is the composite operator $N = JSF$, of the Nemytskii operator F,

$$F : C(I;\mathbf{R}^n) \to C(I;\mathbf{R}^n), \quad F(u)(t) = f(t, u(t)),$$

of a linear integral operator S, of the form

$$S : C(I;\mathbf{R}^n) \to C^1(I;\mathbf{R}^n), \quad S(u)(t) = \int_0^T k(t,s)\, u(s)\, ds$$

and of the imbedding map J,

$$J : C^1(I;\mathbf{R}^n) \to C(I;\mathbf{R}^n), \quad J(u) = u.$$

For the (IVP), the kernel k has the expression

$$k(t,s) = \begin{cases} 1, & s < t \\ 0, & t < s \end{cases}$$

while for the (BVP), $-k$ is the Green's function corresponding to the boundary conditions $\mathcal{B}$, assuming its existence. Assume F and S are bounded continuous operators. Then, since by the Ascoli-Arzèla Theorem, the imbedding map J is completely continuous, we have that N is completely continuous and so, we may think to apply Schauder's Fixed Point Theorem or the Leray-Schauder Principle (see [7]) in order to guarantee the existence of solutions to each of problems (1).

2. Equations in Banach spaces

STEP II: Consider the problems (1) in a Banach space E.

The imbedding map J of $C^1(I;E)$ into $C(I;E)$ is not completely continuous when E is infinite dimensional. Consequently, to say something about the compactness of N, for each bounded set C of $C(I;E)$ we have to analyze the compactness of the section sets $N(C)(t)$ for $t \in I$, where

$$N(C)(t) = \left\{ \int_0^T k(t,s) f(s,u(s))\, ds : \ u \in C \right\}.$$

If C is countable, then the integral and the Kuratowski's measure of noncompactness interchange as follows (see [3], Theorem 1.2.2):

$$\alpha(N(C)(t)) \le \int_0^T |k(t,s)|\, \alpha(f(s,C(s)))\, ds.$$

Next we require the following compactness property holds for f :

$$\alpha(f(t,M)) \le L(t)\, \alpha(M)$$

for each bounded set $M \subset E$. Then we obtain

$$\alpha(N(C)(t)) \le \int_0^T |k(t,s)|\, L(s)\, \alpha(C(s))\, ds.$$

From, we would like to derive that

$$\alpha(N(C)(t)) = 0, \quad \text{for all } t \in I.$$

This is not easy for general sets C, but it is possible if C satisfies

$$C \subset \operatorname{conv}(\{u_0\} \cup N(C))$$

for some $u_0 \in C(I;E)$. Indeed, for such a set C, we have

$$\alpha(C(t)) \le \alpha(N(C)(t)) \le \int_0^T |k(t,s)|\, L(s)\, \alpha(C(s))\, ds.$$

If we let $\phi(t) = \alpha(C(t))$, then

$$\phi(t) \le \int_0^T |k(t,s)|\, L(s)\, \phi(s)\, ds.$$

Now suitable integral inequalities (see [9]) yield $\phi \equiv 0$ and so, by the infinite dimensional version of the Ascoli-Arzèla Theorem, $N(C)$ is relatively compact in $C(I;E)$.

Notice by the above argument we have not proved the complete continuity of N and in consequence, Schauder's Fixed Point Theorem and Leray-Schauder Principle do not apply. However, we may use Mönch's extensions of these two theorems.

3. Mönch's fixed point theorems

Theorem 3.1. *([5]) Let X be a Banach space, $D \subset X$ be closed convex and $N : D \to D$ be continuous with the further property that for some $x_0 \in D$ one has*

$$\left.\begin{array}{c} C \subset D,\ C \ \text{countable}, \\ \overline{C} = \overline{co}\left(\{x_0\} \cup N(C)\right) \end{array}\right\} \Longrightarrow \overline{C} \ \text{compact}. \tag{2}$$

Then N has at least one fixed point.

Theorem 3.2. *([5]) Let X be a Banach space, $K \subset X$ closed convex, $U \subset K$ open in K and $N : \overline{U} \to K$ continuous, with the further property that for some $x_0 \in U$ one has*

$$\left.\begin{array}{c} C \subset \overline{U},\ C \ \text{countable}, \\ C \subset \overline{co}\left(\{x_0\} \cup N(C)\right) \end{array}\right\} \Longrightarrow \overline{C} \ \text{compact}. \tag{3}$$

In addition, assume that

$$x \neq (1-\lambda)\, x_0 + \lambda N(x) \quad \text{for all } x \in \overline{U} \setminus U,\ \lambda \in (0,1).$$

Then N has at least one fixed point in $\overline{U}$.

STEP III: Consider the (IVP) and the (BVP) for a differential inclusion in the Banach space E, i.e.

$$\left\{\begin{array}{l} u' \in f(t,u), \quad t \in I \\ u(0) = 0; \end{array}\right. \qquad \left\{\begin{array}{l} u'' \in f(t,u), \quad t \in I \\ u \in \mathcal{B}. \end{array}\right. \tag{4}$$

If we wish to discuss the inclusions (4) in a similar way like the equations (1), we need to give multivalued analogs to Mönch's Theorems. This was achieved in [6] replacing (2)-(3) by some slightly more general conditions expressed in terms of a pair (M, C) instead of a single set C :

4. Mönch type theorems for inclusions

Theorem 4.1. *([6]) Let D be a closed, convex subset of a Banach space X and $N : D \to 2^D \setminus \{\emptyset\}$ a mapping with convex values. Assume graph(N) is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{c} M \subset D,\ M = conv\left(\{x_0\} \cup N(M)\right), \\ \overline{M} = \overline{C} \ \text{with } C \subset M,\ C \ \text{countable} \end{array}\right\} \Longrightarrow \overline{M} \ \text{compact}. \tag{5}$$

Then there exists $x \in D$ with $x \in N(x)$.

Theorem 4.2. *([6]) Let K be a closed, convex subset of a Banach space X, U a relatively open subset of K and $N : \overline{U} \to 2^K \setminus \{\emptyset\}$ a mapping with convex values.*

Assume graph (N) is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in U$, the following two conditions are satisfied:

$$\left.\begin{array}{l} M \subset \overline{U},\ M \subset \ conv\left(\{x_0\} \cup N(M)\right), \\ \overline{M} = \overline{C} \ \text{ with } C \subset M,\ C \ \text{ countable} \end{array}\right\} \Longrightarrow \overline{M} \ \text{ compact;} \tag{6}$$

$$x \notin (1-\lambda)\, x_0 + \lambda N(x) \quad \text{for all } x \in \overline{U} \setminus U,\ \lambda \in (0,1)\,.$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Notice any upper semicontinuous mapping N with compact convex nonempty values, has closed graph and maps compact sets into relatively compact sets.

5. Hammerstein integral inclusions

Let us present an application of Theorem 4 to the Hammerstein integral inclusion

$$u(t) \in \int_0^T k(t,s)\, f(s, u(s))\, ds \ \text{ a.e. } t \in I. \tag{7}$$

Theorem 5.1. *([8]) Let $p \in [1,\infty]$, $q \in [1,\infty)$ and let $r \in (1,\infty]$ be the conjugate of q, i.e. $1/q + 1/r = 1$. Assume $k : I^2 \to \mathbf{R}$ is measurable and*

$$\left\{\begin{array}{l} (a)\ \text{if } p<\infty: \text{ the map } t \longmapsto k(t,.) \text{ belongs to } L^p\left(I; L^r(I)\right); \\ (b)\ \text{if } p=\infty: \text{ the map } t \longmapsto k(t,.) \text{ belongs to } C\left(I; L^r(I)\right). \end{array}\right.$$

In addition suppose:

(1) *$f : I \times E \to 2^E \setminus \{\emptyset\}$ is a Carathéodory function with compact convex values;*

(2) *there exists $a \in L^q(I; \mathbf{R}_+)$, $b \in \mathbf{R}_+$ and $R > 0$ such that*

$$\left\{\begin{array}{l} (a)\ \text{if } p<\infty: |f(t,x)| \le a(t) + b\,|x|^{p/q},\ x \in E \\ (b)\ \text{if } p=\infty: |f(t,x)| \le a(t) \ \text{ for } |x| \le R \end{array}\right.$$

(i.e. f is a $(q,p/q)$-Carathéodory function);

(3) *there exists a $(q,p/q)$-Carathéodory function $\omega : I \times \mathbf{R}_+ \to \mathbf{R}_+$ with*

$$\alpha\left(f(t,M)\right) \le \omega\left(t, \alpha(M)\right)$$

a.e. $t \in I$, for every bounded $M \subset E$;

(4) *$\varphi \equiv 0$ is the unique solution in $L^p(I; \mathbf{R}_+)$ to the inequality*

$$\varphi(t) \le 2 \int_0^T |k(t,s)|\, \omega(s, \varphi(s))\, ds, \quad \text{a.e. } t \in I;$$

(5) *$|u|_p < R$ for any solution $u \in L^p(I;E)$ with $|u|_p \le R$ of*

$$u(t) \in \lambda \int_0^T k(t,s)\, f(s,u(s))\, ds, \quad \text{a.e. } t \in I,$$

for $\lambda \in (0,1)$.

Then (7) has at least one solution $u \in L^p(I;E)$ (respectively, in $C(I;E)$ if $p = \infty$) with $|u|_p \le R$.

6. Fixed point results for acyclic mappings

STEP IV: Let us now discuss the problems

$$(8)\qquad \begin{cases} u' \in Au + f(t,u), & t \in I \\ u(0) = 0; \end{cases} \qquad \begin{cases} u'' \in Au + f(t,u), & t \in I \\ u \in \mathcal{B}. \end{cases}$$

Notice semilinear parabolic, respectively hyperbolic and elliptic inclusions can be put under the abstract form $u' \in Au + f(t,u)$, respectively $u'' \in Au + f(t,u)$.

Here we suppose that A is a multivalued map from E into 2^E such that for each v in a given space of functions, there exists a unique solution $S(v) := u$ to the initial value problem, respectively boundary value problem:

$$(9)\qquad \begin{cases} u' \in Au + v, & t \in I \\ u(0) = 0; \end{cases} \qquad \begin{cases} u'' \in Au + v, & t \in I \\ u \in \mathcal{B}. \end{cases}$$

We note that the solution operator S is not linear, so even f has convex values, the mapping $N = SF$ may have non convex values. Thus, a natural problem was to give extensions of Mönch's Theorems for multivalued operators with non convex values. As a result we obtained a Mönch type generalization of the Eilenberg-Montgomery Theorem [2] (see also [4]):

Theorem 6.1. *([9]) Let D be a closed convex subset of a Banach space X, Y a metric space, $N : D \to 2^Y \setminus \{\emptyset\}$ a map with acyclic values, and $r : Y \to D$ continuous. Assume graph(N) is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in D$ one has*

$$(10)\qquad \left.\begin{array}{l} M \subset D,\ M = conv\,(\{x_0\} \cup rN(M)), \\ \overline{M} = \overline{C},\quad C \subset M,\ \ C \ \text{countable} \end{array}\right\} \Longrightarrow \overline{M} \ \text{compact.}$$

Then there exists $x \in D$ with $x \in rN(x)$.

The next result is the continuation type version of Theorem 6.

Theorem 6.2. *([9]) Let K be a closed convex subset of a Banach space X, U a convex, relatively open subset of K, Y a metric space, $N : \overline{U} \to 2^Y \setminus \{\emptyset\}$ with acyclic values and $r : Y \to K$ continuous. Assume graph(N) is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in U$, the following two conditions are satisfied:*

$$(11)\qquad \left.\begin{array}{l} M \subset \overline{U},\ M \subset\ conv\,(\{x_0\} \cup rN(M)) \\ \overline{M} = \overline{C},\quad C \subset M,\ \ C \ \text{countable} \end{array}\right\} \Longrightarrow \overline{M} \ \text{compact;}$$

$$(12)\qquad x \notin (1-\lambda)\,x_0 + \lambda rT(x) \quad \text{for all } x \in \overline{U} \setminus U,\ \lambda \in (0,1).$$

Then there exists $x \in \overline{U}$ with $x \in rN(x)$.

7. Abstract Hammerstein inclusions

STEP V: Here we discuss the abstract inclusion

$$u \in SF(u), \quad u \in L^p(I;E), \tag{13}$$

where

$$S : L^q(I;E) \to L^p(I;E)$$

is a given single valued operator and $F : L^p(I;E) \to 2^{L^q(I;E)}$ is the Nemytskii multivalued operator associated to a function $f : I \times E \to 2^E$, given by

$$F(u) = \{w \in L^q(I;E) : w(t) \in f(t, u(t)) \text{ a.e. } t \in I\}.$$

As a direct consequence of Theorem 7, we have the following existence principle for (10).

Theorem 7.1. *([1]) Let K be a closed convex subset of $L^p(I;E)$ $(1 \le p \le \infty)$, U a relatively open subset of K and $u_0 \in U$. Assume*

$(H1)$ $SF : \overline{U} \to 2^K \setminus \{\emptyset\}$ has acyclic values, closed graph and maps compact sets into relatively compact sets;

$(H2)$ $$\left.\begin{array}{l} M \subset \overline{U},\ M \subset \operatorname{conv}(\{0\} \cup SF(M)) \\ \overline{M} = \overline{C},\ C \subset M,\ C \text{ countable} \end{array}\right\} \Longrightarrow \overline{M} \text{ compact;}$$

$(H3)$ $u \notin (1-\lambda)u_0 + \lambda SF(u)$ for all $u \in \overline{U} \setminus U$, $\lambda \in (0,1)$.

Then (10) has at least one solution in $\overline{U}$.

In what follows: $u_0 = 0$, $U = B_R = \{u \in K : |u|_p < R\}$. We shall give sufficient conditions for (H1)-(H2):

(S1) There exists a function $k : I^2 \to R_+$ such that $k(t,.) \in L^r(I)$ $(1/r + 1/q = 1)$, the function $t \longmapsto |k(t,.)|_r$ belongs to $L^p(I)$ and

$$|S(w_1)(t) - S(w_2)(t)| \le \int_I k(t,s)\,|w_1(s) - w_2(s)|\,ds \tag{14}$$

a.e. $t \in I$, for all $w_1, w_2 \in L^q(I;E)$.

(S2) $S : L^q(I;E) \to K$ and for every compact convex subset C of E, S is sequentially continuous from $L^1_w(I;C)$ to $L^p(I;E)$ (Here $L^1_w(I;C)$ stands for the set $L^1(I;C)$ endowed with the weak topology of $L^1(I;E)$).

(f1) $f : I \times E \to 2^E \setminus \{\emptyset\}$ has compact convex values.

(f2) $f(.,x)$ has a strongly measurable selection on I, for each $x \in E$.

(f3) $f(t,.)$ is upper semicontinuous, for a.e. $t \in I$.

(f4) There exists $a \in L^q(I;\mathbf{R}_+)$, $b \in \mathbf{R}_+$ and $R > 0$ such that

$$\begin{cases} \text{if } p < \infty : |f(t,x)| \le a(t) + b\,|x|^{p/q}, \text{ for all } x \in E; \\ \text{if } p = \infty : |f(t,x)| \le a(t), \text{ for } |x| \le R. \end{cases}$$

(f5) For every separable closed subspace E_0 of E, there exists a $(q, p/q)$-Carathéodory function $\omega : I \times R_+ \to R_+$ such that

$$\beta_{E_0}(f(t,M) \cap E_0) \le \omega(t, \beta_{E_0}(M))$$

a.e. $t \in I$, for every set $M \subset E_0$ satisfying

$$|M| \leq |S(0)(t)| + \left(|a|_q + bR^{p/q}\right)|k(t,.)|_r$$

if $p < \infty$, respectively

$$|M| \leq |S(0)(t)| + |a|_q |k(t,.)|_r$$

if $p = \infty$. In addition $\varphi \equiv 0$ is the unique solution in $L^p(I;\mathbf{R}_+)$ to

$$\varphi(t) \leq \int_I k(t,s)\,\omega(s,\varphi(s))\,ds, \quad \text{a.e. } t \in I. \tag{15}$$

Here β_{E_0} is the ball measure of noncompactness in E_0.

(SF) For every $u \in K$ the set $SF(u)$ is acyclic in K.

Theorem 7.2. *([1]) Assume (S1)-(S2), (f1)-(f5) and (SF) hold. In addition suppose (H3). Then (10) has at least one solution u in $K \subset L^p(I;E)$ with $|u|_p \leq R$.*

If $q \leq p$, then a sufficient condition for (f5) is

(f5*) For every separable closed subspace E_0 of E, there exists a $\delta \in L^{pq/(p-q)}(I)$ such that

$$\beta_{E_0}(f(t,M) \cap E_0) \leq \delta(t)\,\beta_{E_0}(M)$$

a.e. $t \in I$, for every subset $M \subset E_0$ satisfying

$$|M| \leq |S(0)(t)| + \left(|a|_q + bR^{p/q}\right)|k(t,.)|_r \ ,$$

if $p < \infty$, respectively

$$|M| \leq |S(0)(t)| + |a|_q |k(t,.)|_r$$

if $p = \infty$, and

$$|\delta|_{pq/(p-q)} \left||k(t,.)|_r\right|_p < 1. \tag{16}$$

Here $pq/(p-q) = q$ if $p = \infty$ and $pq/(p-q) = \infty$ if $p = q$.

Notice in the Volterra case, i.e. when $k(t,s) = 0$ for $s > t$, condition (16) can be dropped.

Example 7.1. *Let $f(t,x) = a|x|^{p-2}x$, where $a > 0$, $p > 2$. Then, if $|M| \leq \eta(t)$, one has*

$$\beta(f(t,M)) \leq a(p-1)\,\eta(t)^{p-2}\,\beta(M).$$

Here $\delta(t) = a(p-1)\,\eta(t)^{p-2}$ and (16) holds for a sufficiently small a.

We note that the technique we use to verify compactness conditions like (5), (6) equally applies to check the Palais-Smale condition in critical point theory (see [10]).

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