Positive solutions of semi-linear elliptic problems via Krasnoselskii type theorems in cones and Harnack's inequality

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ABSTRACT. In this paper a new version of Krasnoselskii's fixed point theorem in cones, together with a global weak Harnack inequality for nonnegative superharmonic functions are used to investigate the existence of positive solutions of the Dirichlet problem for semi-linear elliptic equations.

Key words and phrases: positive solution, fixed point theorem in cones, elliptic boundary value problem, weak Harnack inequality.

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1 Introduction

There exists a huge literature devoted to the existence and localization of positive solutions of various types of integral, ordinary differential and partial differential equations. The most common approaches go from the contraction principle, topological fixed point methods and upper and lower solution techniques, to more advanced methods of degree theory and critical point theory. One of them is based on the Krasnoselskii's fixed point theorem in cones (see [8], [9], [12] and [17]) and has been intensively used in studying boundary value problems for ordinary differential equations (see [1], [2], [5], [6], [10], [13], [14], [17], [19], [24], and [9], [11], [16] for similar results on integral equations). Its success is due to the upper and lower inequalities for the appropriate Green's functions. Similar inequalities for boundary value problems related to partial differential equations are not known and Krasnoselskii's Theorem has appeared quite unapplicable to this type of problems. Some progress in this direction has been made in [21] and [23], where bilateral estimates are used only with respect to one of the variables (say, time variable), or, by iteration, successively, to all of the variables. Obviously, this has required a suitable geometry of the domain of the equation.

The main goal of this paper is to investigate the applicability of Krasnoselskii's Theorem to semi-linear elliptic problems in general domains. The main ingredient will be a global weak Harnack inequality whose use will require a new version of Krasnoselskii's Theorem.

2 Compression-expansion fixed point theorems

Throughout this section E will be a linear space endowed with two norms $|.|$ and $\|.\|$ such that

$$|x| \leq c\,\|x\| \quad \text{for all } x \in E \tag{2.1}$$

and some constant $c > 0$. Also $C \subset E$ will be a *cone*, i.e., a closed convex set with

$$\lambda C \subset C \text{ for all } \lambda \geq 0,\ C \cap (-C) = \{0\} \text{ and } C \neq \{0\}\,.$$

As usual, the notation $x \leq y$ for $x, y \in E$ will stand for $y - x \in C$.

For two numbers r, R with $0 < cr < R$ we shall denote

$$C_{r,R} = \{x \in C : r \leq \|x\|\,,\ |x| \leq R\}\,.$$

Notice $C_{r,R}$ is bounded with respect to norm $|.|$, but can be unbounded with respect to $\|.\|$.

In this paper, a continuous map $N : X \to Y$, where X, Y are topological spaces, is said to be *compact* if $N(X)$ is contained in a compact subset of Y. If X is a metric space, then N is said to be *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y.

Theorem 2.1 *Assume $0 < cr < R$, the map $N : C_{r,R} \to C$ is completely continuous with respect to the $\|.\|$-topology and $N(C_{r,R})$ is bounded with respect to norm $\|.\|$. In addition assume that the following conditions are satisfied:*

(h1) $N(x) \nleq x$ *for all* $x \in C$ *with* $\|x\| = r$,

(h2) $N(x) \ngeq x$ *for all* $x \in C$ *with* $|x| = R$.

Then N has at least one fixed point in C with

$$r < \|x\| \quad \textit{and} \quad |x| < R.$$

Proof. Following the ideas from [20], we consider the map $N' : C \to C$ defined by

$$N'(x) = \begin{cases} \delta h & \text{if } x = 0 \\ \frac{\|x\|}{r} N\left(\frac{r}{\|x\|}x\right) + \delta h & \text{if } 0 < \|x\| \le r - \delta \\ \frac{\|x\|}{r} N\left(\frac{r}{\|x\|}x\right) + h\left(r - \|x\|\right) & \text{if } r - \delta \le \|x\| \le r \\ N(x) & \text{if } x \in C_{r,R} \\ N\left(\frac{R}{|x|}x\right) & \text{if } R \le |x| \end{cases}$$

where $0 < \delta < r$ and $h \in C \setminus \{0\}$. Notice the values of N' only depend on the values of N in $C_{r,R}$. Let

$$r_0 = \sup\left\{\left\|N'(x)\right\| : x \in C\right\}.$$

Then, in particular,

$$\left\|N'(x)\right\| \le r_0 \text{ for every } x \in C \text{ with } \|x\| \le r_0.$$

Also N' is compact on $\{x \in C : \|x\| \le r_0\}$ with respect to norm $\|.\|$. From the Schauder fixed point theorem we have that there exists a point $x \in C$ with

$$N'(x) = x \text{ and } \|x\| \le r_0.$$

It remains to show that $x \in C_{r,R}$. Clearly $x \neq 0$. Assume $0 < \|x\| \le r - \delta$. Then

$$\frac{\|x\|}{r} N\left(\frac{r}{\|x\|}x\right) + \delta h = x$$

whence

$$\frac{r}{\|x\|}x \ge N\left(\frac{r}{\|x\|}x\right)$$

which contradicts (h1). We obtain the same contradiction if we assume $r - \delta \le \|x\| \le r$. Finally assume $R \le |x|$. Then

$$N\left(\frac{R}{|x|}x\right) = x = \frac{R}{|x|}x + \left(1 - \frac{R}{|x|}\right)x \ge \frac{R}{|x|}x$$

which contradicts (h2). Therefore $x \in C_{r,R}$ as wished. ■

Theorem 2.2 *Assume $0 < cr < R$, the map $N : C_{r,R} \to C$ is compact with respect to the $\|.\|$-topology, and that there is a constant c_1 such that*

$$\left\|\frac{1}{\lambda} N(\lambda x)\right\| \le c_1 \text{ whenever } x \in C_{r,R},\ \lambda > 0 \text{ and } \lambda x \in C_{r,R}. \tag{2.2}$$

In addition assume that the following conditions are satisfied:

(h1) $N(x) \ngeq x$ *for all* $x \in C$ *with* $\|x\| = r$,

(h2) $N(x) \nleq x$ *for all* $x \in C$ *with* $|x| = R$.

Then N *has at least one fixed point in* C *with*

$$r < \|x\| \quad \textit{and} \quad |x| < R.$$

Proof. Let $N' : C_{r,R} \to C$ be given by

$$N'(x) = \left(\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \right)^{-1} N\left(\left(\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \right) x \right).$$

Notice N' is well defined since

$$\left(\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \right) x \in C_{r,R} \quad \text{for all } x \in C_{r,R}.$$

Also note if $\|x\| \leq M$, then

$$\left(\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \right)^{-1} \leq \frac{M}{r}.$$

Consequently, N' sends bounded sets into relatively compact sets (with respect to the $\|.\|$-topology). Hence N' is completely continuous. Also (2.2) guarantees that $N'(C_{r,R})$ is bounded with respect to norm $\|.\|$ (note that $\left(\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \right)^{-1}$ may tends to infinity as $|x| \to R$ and $\|x\| \to \infty$). Now observe that N' satisfies all the assumptions of Theorem 2.1. Hence N' has a fixed point $y \in C_{r,R}$. Obviously,

$$x := \left(\frac{R}{|y|} + \frac{r}{\|y\|} - 1 \right) y$$

is a fixed point of N. ■

Remark 2.3 *In case that there exists a constant* $c_0 > 0$ *with*

$$c_0 \|x\| \leq |x| \quad \textit{for all } x \in C, \tag{2.3}$$

condition (2.2) *trivially holds since*

$$\frac{R}{|x|} + \frac{r}{\|x\|} - 1 \geq \frac{c_0 r}{R} > 0.$$

Under assumption (2.3) (see also [22] and [18]), Theorems 2.1 and 2.2 yield the following result:

Theorem 2.4 *Assume* (2.3)*,* $0 < cr < R$ *and* $N : C_{r,R} \to C$ *is a compact map. In addition assume that one of the following two conditions holds:*

(i) $N(x) \nleq x$ *for all* $x \in C$ *with* $\|x\| = r$ *and* $N(x) \ngeq x$ *for all* $x \in C$ *with* $|x| = R$;

(ii) $N(x) \ngeq x$ *for all* $x \in C$ *with* $\|x\| = r$ *and* $N(x) \nleq x$ *for all* $x \in C$ *with* $|x| = R$.

Then N *has at least one fixed point* $x \in C$ *with*

$$r < \|x\| \quad \textit{and} \quad |x| < R.$$

3 Application to boundary value problems

We consider the Dirichlet problem for a semi-linear elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where Ω is a bounded regular domain in $\mathbf{R}^n$, $n \geq 1$, $f : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous and $f(u) = f \circ u$. We seek positive solutions (see also [3], [4] and [15]), i.e., $u \in C^1\left(\overline{\Omega}\right)$, $u(x) > 0$ for all $x \in \Omega$ and u satisfies (3.1), where Δu is considered in the sense of distributions.

The main assumption will be the a global weak Harnack inequality for nonnegative superharmonic functions. By a *superharmonic* function in a domain $\Omega \subset \mathbf{R}^n$ we mean a function $u \in C^1(\Omega)$ with

$$\Delta u \leq 0 \ \text{ in } \ \mathcal{D}'(\Omega),$$

that is,

$$\int_\Omega \nabla u \cdot \nabla v \geq 0 \ \text{ for every } v \in C_0^\infty(\Omega) \text{ satisfying } v(x) \geq 0 \text{ on } \Omega.$$

We shall assume that the following *global weak Harnack inequality* holds:

(H) there exists $p \in [1, \infty]$, a compact set $K \subset \Omega$ and a number $\eta > 0$ such that

$$u(x) \geq \eta\, |u|_{L^p(\Omega)} \ \text{ for all } x \in K$$

and every nonnegative superharmonic function $u \in C^1\left(\overline{\Omega}\right)$ with $u = 0$ on $\partial\Omega$.

In order to apply Theorem 2.2, let

$$E = C_0\left(\overline{\Omega}\right) := \left\{u \in C\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\right\}$$

be endowed with norms $\|.\| = |.|_\infty$ and $|.| = |.|_p$, where

$$|u|_\infty = \max_{x\in\overline{\Omega}} |u(x)|, \qquad |u|_p = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}.$$

Clearly (2.1) holds with $c = (\mu(\Omega))^{\frac{1}{p}}$, where $\mu(\Omega)$ is the measure of Ω.

Let

$$C = \left\{u \in C_0\left(\overline{\Omega}; \mathbf{R}_+\right) : u(x) \geq \eta\, |u|_{L^p(\Omega)} \text{ for all } x \in K\right\}.$$

Define

$$N : C\left(\overline{\Omega}; \mathbf{R}_+\right) \to C\left(\overline{\Omega}\right) \text{ by } N(u) = (-\Delta)^{-1} F(u),$$

where

$$F : C\left(\overline{\Omega}; \mathbf{R}_+\right) \to C\left(\overline{\Omega}\right), \quad F(u)(x) = f(u(x)).$$

Since $f \geq 0$, and $(-\Delta)^{-1}$ is positive, we have that N maps the set $C\left(\overline{\Omega}; \mathbf{R}_+\right)$ into itself. Also, by the global weak Harnack inequality, condition (H), we have $N(C) \subset C$.

Next we assume

(C) There exists $c_0 > 0$ and $\theta \in [1, \frac{2p}{n}) \cap [1, p]$, such that

$$f(\tau) \leq c_0 \tau^\theta \text{ for all } \tau \in \mathbf{R}_+.$$

This condition guarantees that for every $R > 0$, the restriction of N to $C_R := \left\{u \in C : |u|_p \leq R\right\}$ is compact with respect to the $|.|_\infty$-topology, and that there exists a $c_1 > 0$ such that

$$\left|\frac{1}{\lambda} N(\lambda u)\right|_\infty \leq c_1 \text{ whenever } u \in C, \; |u|_p \leq R \text{ and } \lambda \in (0,1). \tag{3.2}$$

Indeed, if we denote $q = \frac{p}{\theta}$, then we have that $F(C_R)$ is bounded in $L^q(\Omega)$, whence $N(C_R)$ is bounded in $W^{2,q}(\Omega)$. Since $q > \frac{n}{2}$, we deduce that $N(C_R)$ is relatively compact in $C\left(\overline{\Omega}\right)$. Hence the restriction of N to C_R is compact with respect to the $|.|_\infty$-topology.

Furthermore, we have

$$\begin{aligned}\left|\frac{1}{\lambda}N(\lambda u)\right|_\infty &\le \gamma\left|\frac{1}{\lambda}N(\lambda u)\right|_{W^{2,q}(\Omega)}\\ &\le \gamma\frac{1}{\lambda}\left|(-\Delta)^{-1}\right| |F(\lambda u)|_q\\ &\le \gamma\lambda^{\theta-1}\left|(-\Delta)^{-1}\right| c_0 |u|_p^\theta\\ &\le \gamma\left|(-\Delta)^{-1}\right| c_0 R^\theta =: c_1\end{aligned}$$

which shows (3.2).

Now assume that the following condition is satisfied:

(A1) There exists $r > 0$ such that

$$\frac{\max_{\tau\in[0,r]} f(\tau)}{r} < \left|(-\Delta)^{-1}1\right|_\infty^{-1}.$$

Then, if $u \in C$, $|u|_\infty = r$ and $N(u) \ge u$, since

$$f(u(x)) \le \max_{\tau\in[0,r]} f(\tau),$$

we derive

$$r = |u|_\infty \le |N(u)|_\infty \le \left|(-\Delta)^{-1}1\right|_\infty \max_{\tau\in[0,r]} f(\tau) < r,$$

a contradiction. Hence (A1) guarantees (h1) in Theorem 2.2.

Finally, assume:

(A2) There exists $R > cr$ such that

$$\frac{\inf_{\tau\in[\eta R,\infty)} f(\tau)}{R} > \left|(-\Delta)^{-1}(1|_K)\right|_p^{-1}.$$

Here by $h|_K$ we have denoted the function defined as $h(x)$ for $x \in K$, $h(x) = 0$ for $x \in \Omega \setminus K$.

Let $u \in C$, $|u|_p = R$ and $N(u) \le u$. From

$$F(u)(x) = f(u(x)) \ge \inf_{\tau\in[\eta R,\infty)} f(\tau), \quad \text{for all } x \in K,$$

we derive

$$\begin{aligned} R &= |u|_p \geq |N(u)|_p \\ &\geq \left| (-\Delta)^{-1} \left(F(u)|_K \right) \right|_p \\ &\geq \left| (-\Delta)^{-1} \left(1|_K \right) \right|_p \inf_{\tau \in [\eta R, \infty)} f(\tau) \\ &> R \end{aligned}$$

a contradiction. Hence (A2) guarantees (h2) in Theorem 2.2.

Thus we have obtained the following result.

Theorem 3.1 *Under assumptions* (H), (C), (A1) *and* (A2), *problem* (3.1) *has a positive solution* u *satisfying*

$$|u|_\infty > r \quad \text{and} \quad |u|_p < R.$$

Remark 3.2 *If* f *is nondecreasing on* $\mathbf{R}_+$, *then conditions* (A1), (A2) *become*

$$\frac{f(r)}{r} < \left| (-\Delta)^{-1} 1 \right|_\infty^{-1} \tag{3.3}$$

and

$$\frac{f(\eta R)}{\eta R} > \eta^{-1} \left| (-\Delta)^{-1} \left(1|_K \right) \right|_p^{-1} \tag{3.4}$$

respectively. In this case, the existence of the two numbers r, R *satisfying* (3.3), (3.4) *and* $cr < R$, *is guaranteed by the following behavior of* f *at zero and infinity:*

$$\liminf_{\tau \to 0} \frac{f(\tau)}{\tau} < \left| (-\Delta)^{-1} 1 \right|_\infty^{-1}$$

and

$$\limsup_{\tau \to \infty} \frac{f(\tau)}{\tau} > \eta^{-1} \left| (-\Delta)^{-1} \left(1|_K \right) \right|_p^{-1}.$$

Also, multiple solutions to (3.1) *can be obtained if inequalities* (3.3) *and* (3.4) *are satisfied for several pairs of numbers* (r, R).

We finish this section by some comments about Harnack inequalities.

For $n = 1$, inequality (H) holds for every $p \in [1, \infty]$. This follows since every superharmonic function in a real interval is concave. Now if $\Omega = (a, b)$, $K = [c, d]$, $a < c < d < b$, and $u \in C^1[a, b]$ is nonnegative, superharmonic in (a,b), $u(a) = u(b) = 0$, and $\min_{x \in [c,d]} u(x) = 1$, then, if $u(x_0) = 1$ for some $x_0 \in [c,d]$, the concavity of u implies that

$$u(x) \leq (d-b)^{-1} (x-b) \quad \text{for all } x \in [a, x_0]$$

and

$$u(x) \leq (c-a)^{-1} (x-a) \quad \text{for all } x \in [x_0, b].$$

Hence there exists a constant $\delta > 0$ only depending on a, b, c and d, with

$$u(x) \leq \delta \text{ for all } x \in [a,b].$$

Then (H) holds with $p = \infty$ and $\eta = \delta^{-1}$.

For $n \geq 2$, by our knowledge, inequalities of type (H) are not known. However, some partial results come to support (H) as a **conjecture**, at least for some suitable values of p. Indeed, if we denote by $B_\rho(x)$ the open ball in $\mathbf{R}^n$ with centre x and radius ρ and abbreviate $B_\rho(x) = B_\rho$ when the centre is not important, we have the following basic result (see [7, Theorem 8.18]):

Proposition 3.3 (weak Harnack inequality) *Let $\rho > 0$, $n \geq 3$ and $p \in [1, \frac{n}{n-2})$. There exists a constant $\eta_0 > 0$ such that for every nonnegative superharmonic function u in $B_{4\rho}$, the following inequality holds:*

$$u(x) \geq \eta_0 |u|_{L^p(B_{2\rho})} \quad \textit{for all } x \in B_\rho.$$

This proposition yields:

Proposition 3.4 *Let $B_{4\rho} \subset \Omega$, $n \geq 3$, $p \in [1, \frac{n}{n-2})$ and $\Omega_0 \subset\subset \Omega$. There exists a constant $\eta_1 > 0$ such that for every nonnegative superharmonic function u in Ω, the following inequality holds:*

$$u(x) \geq \eta_1 |u|_{L^p(\Omega_0)} \quad \textit{for all } x \in B_\rho.$$

Therefore, an open problem is to investigate property (H), in terms of p and of the regularity of Ω.

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