Positive Solutions of Nonlinear Systems via the Vector Version of Krasnoselskii's Fixed Point Theorem in Cones

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Abstract

The vector version of Krasnoselskii's fixed point theorem in cones is used to obtain existence and localization results for the Dirichlet boundary value problem associated to second order ordinary differential systems with nonlinearities having different behaviors both in components and variables.

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1 Introduction

In our recent paper [6], we have presented a vector version of Krasnoselskii's fixed point theorem in cones (see [1], [2], [3]) by which a solution of a system of equations has been localized in a vector conical shell. Our approach was inspired by the work of Perov and Kibenko [4], where a vector version of the contraction principle was given for the treatment of a system of equations. The same idea was used in [5] for a vector version of the continuation principle for contractions. The use of a vector norm in an abstract existence principle, when applied to systems, makes possible that the nonlinear term of a system may have different behaviors both in components and variables. The aim of this paper is to illustrate the applicability of the above principle to the Dirichlet boundary value problem for second order ordinary differential systems.

We shall use the following notions and notations. If $(X, |.|)$ is a normed linear space, by a cone of X we mean a closed convex subset $K \subset X$ with $K \setminus \{0\} \neq \emptyset$, $\lambda K \subset K$ for every $\lambda \in \mathbf{R}_+$ and $K \cap (-K) = \{0\}$. Any cone K induces a partial order relation in X, denoted by $\preceq$. Hence $u \preceq v$ if and only if $v - u \in K$. We shall say that $u \prec v$ if $v - u \in K \setminus \{0\}$.

In what follows, we shall consider two cones K_1, K_2 of X, the corresponding cone $K := K_1 \times K_2$ of X^2, and we shall use the same symbol $\preceq$ to denote the partial order relations induced by K in X^2, and by K_1, K_2 in X. Similarly, the same symbol $\prec$ will be used to denote the strict order relations induced by K_1 and K_2 in X. Also, in X^2, the symbol $\prec$ will have the following meaning: $u \prec v$ $(u, v \in X^2)$ if $u_i \prec v_i$ for $i = 1, 2$. For $r, R \in \mathbf{R}_+^2$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we write $0 < r < R$ if $0 < r_1 < R_1$ and $0 < r_2 < R_2$, and we use the notations:

$$\begin{aligned} (K_i)_{r_i,R_i} \quad &: \quad = \{u \in K_i : r_i \leq |u| \leq R_i\} \quad (i = 1, 2) \\ K_{r,R} \quad &: \quad = \{u \in K : r_i \leq |u_i| \leq R_i \text{ for } i = 1, 2\}. \end{aligned}$$

Clearly, $K_{r,R} = (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$.

We are interested in the existence of a solution $u := (u_1, u_2)$ to the operator system

$$\begin{cases} u_1 = N_1(u_1, u_2) \\ u_2 = N_2(u_1, u_2) \end{cases}$$

in the vector conical shell $K_{r,R}$, more exactly with $u \in K$ and

$$r_1 \leq |u_1| \leq R_1, \quad r_2 \leq |u_2| \leq R_2.$$

The main result in [6] is the following vector version of Krasnoselskii's fixed point theorem in cones:

Theorem 1.1 *Let $(X, |.|)$ be a normed linear space; $K_1, K_2 \subset X$ two cones; $K := K_1 \times K_2$; $r, R \in \mathbf{R}_+^2$ with $0 < r < R$, and $N : K_{r,R} \to K$, $N = (N_1, N_2)$ a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) *$N_i(u) \nprec u_i$ if $|u_i| = r_i$, and $N_i(u) \nsucc u_i$ if $|u_i| = R_i$;*

(b) *$N_i(u) \nsucc u_i$ if $|u_i| = r_i$, and $N_i(u) \nprec u_i$ if $|u_i| = R_i$.*

Then N has a fixed point u in K with $r_i \leq |u_i| \leq R_i$ for $i \in \{1, 2\}$.

Remark 1.1 In Theorem 1.1 four cases are possible for $u \in K_{r,R}$:

(c1) $N_1(u) \nprec u_1$ if $|u_1| = r_1$, $N_1(u) \nsucc u_1$ if $|u_1| = R_1$,
$N_2(u) \nprec u_2$ if $|u_2| = r_2$, $N_2(u) \nsucc u_2$ if $|u_2| = R_2$;

(c2) $N_1(u) \nprec u_1$ if $|u_1| = r_1$, $N_1(u) \nsucc u_1$ if $|u_1| = R_1$,
$N_2(u) \nsucc u_2$ if $|u_2| = r_2$, $N_2(u) \nprec u_2$ if $|u_2| = R_2$;

(c3) $N_1(u) \nsucc u_1$ if $|u_1| = r_1$, $N_1(u) \nprec u_1$ if $|u_1| = R_1$,
$N_2(u) \nprec u_2$ if $|u_2| = r_2$, $N_2(u) \nsucc u_2$ if $|u_2| = R_2$;

(c4) $N_1(u) \nsucc u_1$ if $|u_1| = r_1$, $N_1(u) \nprec u_1$ if $|u_1| = R_1$,
$N_2(u) \nsucc u_2$ if $|u_2| = r_2$, $N_2(u) \nprec u_2$ if $|u_2| = R_2$.

2 Positive solutions of the Dirichlet boundary value for systems

We investigate the existence and localization of positive solutions of the nonlinear differential system

$$\left\{\begin{array}{l} u_1''(t)+f_1(t,u_1(t),u_2(t))=0 \\ u_2''(t)+f_2(t,u_1(t),u_2(t))=0 \end{array}\right. \tag{2.1}$$

for $t\in[0,1]$, with

$$u_i(0)=u_i(1)=0,\ \ i=1,2. \tag{2.2}$$

Here $f_i\in C\left([0,1]\times\mathbf{R}_+^2,\mathbf{R}_+\right)$ for $i\in\{1,2\}$.

We seek positive solutions to system (2.1), i.e., pairs $u:=(u_1,u_2)$ of functions from $C\left([0,1],\mathbf{R}_+\right)$ which satisfy (2.1)-(2.2).

Let $X=C[0,1]$ be endowed with norm $|v|_\infty=\max_{t\in[0,1]}|v(t)|$, and let P be the cone of all nonnegative functions from X. Let

$$G(t,s)=\left\{\begin{array}{ll} t(1-s) & \text{if } 0\le t\le s\le 1 \\ s(1-t) & \text{if } 0\le s\le t\le 1 \end{array}\right.$$

be the Green function associated to the differential operator $-u''$ and the Dirichlet boundary condition. Now the problem of finding nonnegative solutions for (2.1)-(2.2) is equivalent to the integral system in P^2,

$$\left\{\begin{array}{l} u_1(t)=\int_0^1 G(t,s)f_1(s,u_1(s),u_2(s))\,ds \\ u_2(t)=\int_0^1 G(t,s)f_2(s,u_1(s),u_2(s))\,ds. \end{array}\right. \tag{2.3}$$

Let $N:P^2\to P^2$ be the completely continuous map $N=(N_1,N_2)$ given by

$$N_i(u)(t)=\int_0^1 G(t,s)f_i(s,u_1(s),u_2(s))\,ds,\ \ i=1,2.$$

Then (2.3) is equivalent to the fixed point problem

$$u=N(u),\ \ u\in P^2.$$

Now we fix any sybinterval $[a,b]$ of $[0,1]$, with $0<a<b<1$, and we easily check that

$$\begin{array}{rcl} G(t,s) & \le & G(s,s)\ \text{for all } t,s\in[0,1]\ \text{and} \\ MG(s,s) & \le & G(t,s)\ \text{for } t\in[a,b],\ s\in[0,1] \end{array} \tag{2.4}$$

where $M=\min\{a,1-b\}$.

If $v\in P$,

$$u(t):=\int_0^1 G(t,s)v(s)\,ds$$

and $u(t_0) = |u|_\infty$, then according to (2.4), for every $t \in [a,b]$, we have

$$\begin{aligned} u(t) &\geq M\int_0^1 G(s,s)\,v(s)\,ds \geq M\int_0^1 G(t_0,s)\,v(s)\,ds \\ &= Mu(t_0) = M\,|u|_\infty . \end{aligned}$$

Thus, if in $X := C[0,1]$ we consider the cone $K_1 = K_2$, defined as

$$K_1 := \{v \in P : \ v(t) \geq M\,|v|_\infty \ \text{for all } t \in [a,b]\}$$

and in X^2 the corresponding cone $K := (K_1)^2$, then we have that $N(K) \subset K$.

Now it is clear to what cone and operator we intend to apply Theorem 1.1.

Before we state our main result, we introduce the following notations. For $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ we let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1,2)$, and

$$\begin{aligned} \gamma_1 &= \min\{f_1(t,u_1,u_2) : \ a \leq t \leq b,\ M\beta_1 \leq u_1 \leq \beta_1,\ Mr_2 \leq u_2 \leq R_2\} \\ \gamma_2 &= \min\{f_2(t,u_1,u_2) : \ a \leq t \leq b,\ Mr_1 \leq u_1 \leq R_1,\ M\beta_2 \leq u_2 \leq \beta_2\} \\ \Gamma_1 &= \max\{f_1(t,u_1,u_2) : \ 0 \leq t \leq 1,\ 0 \leq u_1 \leq \alpha_1,\ 0 \leq u_2 \leq R_2\} \\ \Gamma_2 &= \max\{f_2(t,u_1,u_2) : \ 0 \leq t \leq 1,\ 0 \leq u_1 \leq R_1,\ 0 \leq u_2 \leq \alpha_2\}. \end{aligned}$$

Also, let

$$B = \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds, \quad A = \min_{t\in[a,b]} \int_a^b G(t,s)\,ds.$$

Clearly, $A > 0$ and by direct computation, $B = \frac{1}{8}$.

Theorem 2.1 *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1,2$, such that*

$$\begin{aligned} B\Gamma_1 &< \alpha_1, \quad A\gamma_1 > \beta_1 \\ B\Gamma_2 &< \alpha_2, \quad A\gamma_2 > \beta_2. \end{aligned} \tag{2.5}$$

Then (2.1)-(2.2) *has a positive solution $u = (u_1,u_2)$ with $r_i \leq |u_i|_\infty \leq R_i$, $i = 1,2$, where $r_i = \min\{\alpha_i,\beta_i\}$, $R_i = \max\{\alpha_i,\beta_i\}$. Moreover, the orbit of u for $t \in [a,b]$ is included in the rectangle $[Mr_1, R_1] \times [Mr_2, R_2]$.*

Proof. First note that if $u \in K_{r,R}$, then $r_1 \leq |u_1|_\infty \leq R_1$ and $r_2 \leq |u_2|_\infty \leq R_2$, and by the definition of K,

$$Mr_1 \leq u_1(t) \leq R_1 \text{ and } Mr_2 \leq u_2(t) \leq R_2$$

for all $t \in [a,b]$, showing that the orbit of u for $t \in [a,b]$ is included in the rectangle $[Mr_1, R_1] \times [Mr_2, R_2]$. Also, if we know for example that $|u_1|_\infty = \alpha_1$, then $u_1(t) \leq \alpha_1$ for all $t \in [0,1]$ and

$$M\alpha_1 \leq u_1(t) \leq \alpha_1 \ \text{ for all } t \in [a,b].$$

We claim that for every $u \in K_{r,R}$ and $i \in \{1,2\}$, the following properties hold:

$$\begin{aligned} |u_i|_\infty &= \alpha_i \text{ implies } u_i \not\prec N_i(u) \\ |u_i|_\infty &= \beta_i \text{ implies } u_i \not\succ N_i(u) \end{aligned} \tag{2.6}$$

guaranteeing the applicability of Theorem 1.1.

Indeed, if $|u_1|_\infty = \alpha_1$ and we would have that $u_1 \prec N_1(u)$, then

$$u_1(t) \leq N_1(u)(t) \leq \Gamma_1 \int_0^1 G(t,s)\, ds \leq B\Gamma_1$$

for all $t \in [0,1]$. This yields the contradiction $\alpha_1 < \alpha_1$. Now if $|u_1|_\infty = \beta_1$ and $u_1 \succ N_1(u)$, then for $t \in [a,b]$, we obtain

$$u_1(t) \geq N_1(u)(t) \geq \int_a^b G(t,s)\, f_1(s, u_1(s), u_2(s))\, ds \geq A\gamma_1$$

whence, in particular, we deduce $\beta_1 > \beta_1$, a contradiction. Hence (2.6) holds for $i = 1$. Similarly, (2.6) is true for $i = 2$. ■

In particular, if f_1 and f_2 do not depend on t, i.e., $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$, and f_1, f_2 have some monotonicity properties in u_1 and u_2, for $u_1 \in [0, R_1]$ and $u_2 \in [0, R_2]$, then we can precise the numbers $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$. For example:

(1) if f_1, f_2 are nondecreasing in u_1 and u_2, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, R_2)\ , \quad \gamma_1 = f_1(M\beta_1, Mr_2) \\ \Gamma_2 &= f_2(R_1, \alpha_2)\ , \quad \gamma_2 = f_2(Mr_1, M\beta_2)\,. \end{aligned}$$

(2) if f_1 is nondecreasing in u_1 and u_2, f_2 is nondecreasing in u_1 and nonincreasing in u_2, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, R_2)\,, \quad \gamma_1 = f_1(M\beta_1, Mr_2) \\ \Gamma_2 &= f_2(R_1, 0)\,, \quad \gamma_2 = f_2(Mr_1, \beta_2)\,. \end{aligned}$$

(3) if f_1 is nondecreasing in u_1 and nonincreasing in u_2, f_2 is nonincreasing in u_1 and nondecreasing in u_2, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, 0)\,, \quad \gamma_1 = f_1(M\beta_1, R_2) \\ \Gamma_2 &= f_2(0, \alpha_2)\,, \quad \gamma_2 = f_2(R_1, M\beta_2)\,. \end{aligned}$$

Notice that conditions (2.5) indicate the behavior of f_1, f_2 in some regions of $\mathbf{R}_+^2$, in order to establish the existence and the localization of at least one solution. Combined with monotonicity properties like those in (1)-(3), the hypothesis (2.5) show us how the nonlinearities f_1, f_2 behave at four points in $\mathbf{R}_+^2$. Also, it is clear that by imposing further conditions of this type we can obtain multiple solutions results.

Example 2.1 Let $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ be nondecreasing in u_1 and u_2 for $u_1, u_2 \in R_+$. If

$$\lim_{x\to\infty} \frac{f_i(x,x)}{x} = 0 \text{ and } \lim_{x\to 0} \frac{f_i(x,x)}{x} = \infty \tag{2.7}$$

for $i \in \{1,2\}$, then (2.1)-(2.2) has a positive solution.

Indeed, from (2.7) there are α_1, β_1 with $0 < \beta_1 < \alpha_1$ such that

$$\frac{f_i(\alpha_1, \alpha_1)}{\alpha_1} < \frac{1}{B}, \quad \frac{f_i(M\beta_1, M\beta_1)}{M\beta_1} > \frac{1}{MA} \tag{2.8}$$

for $i \in \{1,2\}$. Let $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. Then $r_i = \beta_1, R_i = \alpha_1$, and according to (1), $\Gamma_i = f_i(\alpha_1, \alpha_1)$, $\gamma_i = f_i(M\beta_1, M\beta_1)$ for $i \in \{1,2\}$. Now (2.8) guarantees (2.5).

Example: $f_i(x,y) = (xy)^{\frac{1}{3}}$, $i = 1, 2$.

Example 2.2 Let $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ be nondecreasing in u_1 and u_2 for $u_1, u_2 \in R_+$. Assume that

$$\lim_{x\to\infty} \frac{f_2(x,x)}{x} = 0, \quad \lim_{x\to 0} \frac{f_2(x,x)}{x} = \infty, \tag{2.9}$$

$$\lim_{x\to\infty} \frac{f_1(x,0)}{x} = \infty \text{ and } \lim_{x\to 0} \frac{f_1(x,y)}{x} = 0 \text{ for every } y > 0. \tag{2.10}$$

Then (2.1)-(2.2) has a positive solution.

Indeed, from (2.9) there are $\alpha_0, \beta_0 > 0$ such that

$$\frac{f_2(M\alpha_1, M\alpha_1)}{M\alpha_1} > \frac{1}{MA} \text{ and } \frac{f_2(\beta_1, \beta_1)}{\beta_1} < \frac{1}{B} \tag{2.11}$$

for every $\alpha_1 \le \alpha_0$ and $\beta_1 \ge \beta_0$. Let $\alpha_1 < \beta_1$, $\alpha_2 = \beta_1$ and $\beta_2 = \alpha_1$. Then $r_i = \alpha_1$, $R_i = \beta_1$ for $i \in \{0,1\}$, and according to (1), $\Gamma_1 = f_1(\alpha_1, \beta_1)$, $\Gamma_2 = f_2(\beta_1, \beta_1)$, $\gamma_1 = f_1(M\beta_1, M\alpha_1)$ and $\gamma_2 = f_2(M\alpha_1, M\alpha_1)$. Clearly (2.11) guarantees that the inequalities in (2.5) corresponding to $i = 2$ hold for every $\alpha_1 \le \alpha_0$ and $\beta_1 \ge \beta_0$. Now due to (2.10), since

$$\frac{f_1(M\beta_1, M\alpha_1)}{M\beta_1} \ge \frac{f_1(M\beta_1, 0)}{M\beta_1}$$

we may first choose $\beta_1 \ge \beta_0$ with $\frac{f_1(M\beta_1,0)}{M\beta_1} > \frac{1}{MA}$, and then $\alpha_1 \le \alpha_0$, $0 < \alpha_1 < \beta_1$ with $\frac{f_1(\alpha_1,\beta_1)}{\alpha_1} < \frac{1}{B}$. Thus condition (2.5) is satisfied.

Example: $f_1(x,y) = x^2(1+y^2)$, $f_2(x,y) = (xy)^{\frac{1}{3}}$.

Example 2.3 Let $f_1(u_1, u_2)$ be nondecreasing in u_1 and nonincreasing in u_2, and let $f_2(u_1, u_2)$ be nonincreasing in u_1 and nondecreasing in u_2, for $u_1, u_2 \in R_+$. Assume that

$$\begin{aligned} \lim_{x\to 0} \frac{f_1(x,0)}{x} &= 0, \quad \lim_{x\to\infty} \frac{f_1(Mx,x)}{x} = \infty \\ \lim_{x\to 0} \frac{f_2(0,x)}{x} &= 0, \quad \lim_{x\to\infty} \frac{f_2(x,Mx)}{x} = \infty. \end{aligned} \tag{2.12}$$

Then (2.1) has a positive ω-periodic solution.

Indeed, from (2.12) it follows that there exist α_1 and β_1 with $0 < \alpha_1 < \beta_1$ such that

$$\begin{aligned} \frac{f_1(\alpha_1, 0)}{\alpha_1} &< \frac{1}{B}, \quad \frac{f_1(M\beta_1, \beta_1)}{\beta_1} > \frac{1}{A} \\ \frac{f_2(0, \alpha_1)}{\alpha_1} &< \frac{1}{B}, \quad \frac{f_2(\beta_1, M\beta_1)}{\beta_1} > \frac{1}{A}. \end{aligned} \tag{2.13}$$

Let $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. Then $r_i = \alpha_1$, $R_i = \beta_1$ for $i \in \{1, 2\}$. Also, by (3), $\Gamma_1 = f_1(\alpha_1, 0)$, $\Gamma_2 = f_2(0, \alpha_1)$, $\gamma_1 = f_1(M\beta_1, \beta_1)$ and $\gamma_2 = f_2(\beta_1, M\beta_1)$. Now (2.13) guarantees (2.5).

Example: $f_1(x, y) = \frac{x^3}{y+1}$, $f_2(x, y) = \frac{y^3}{x+1}$.

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