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Nonresonance Theory for Semilinear Operator Equations under Regularity Conditions

Dezideriu Muzsi (Cluj-Napoca) Radu Precup (Cluj-Napoca)

Abstract. A general nonresonance theory of semilinear operator equations under regularity conditions is developed. Existence of weak solutions (in the energetic space) is established by means of several fixed point principles. Typical applications to elliptic equations with convection terms are presented.

Key Words: nonlinear operator equation, fixed point, nonresonance, eigenvalues, energetic norm, elliptic equation.
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1 Introduction and Preliminaries

In this paper we present existence results for the problem

$$\left\{ \begin{array}{l} Au = cu + F(u, Su) \\ u \in H_A, \end{array} \right. \tag{1.1}$$

◊Dezideriu Muzsi, Department of Applied Mathematics
Babeş–Bolyai University
400084 Cluj-Napoca, Romania, email:
◊Radu Precup, Department of Applied Mathematics
Babeş–Bolyai University
400084 Cluj-Napoca, Romania, email:

where A is a linear and positively defined operator defined on a subspace of a Hilbert space H, the constant c is not an eigenvalue of the operator A and the operator S is a linear and continuous operator

$$S : H_A \to Y,$$

from the energetic space H_A associated to the operator A to a Hilbert space Y, such that

$$\|S\| \leq 1. \tag{1.2}$$

The operator F is a general continuous operator

$$F : H \times Y \to H.$$

The aim of this paper is to extend the results from [9] making them applicable to elliptic equations with convection terms. The technique used in the proof of our results was initiated by Mawhin and Ward in the early eighties ([5], [6]) and since then it has been extensively used for different classes of ordinary differential equations, partial differential equations and integral equations (see, for example [1], [3], [7], [10], [11], [12], [13], [14], [15]). The typical example of problem (1.1) is the following boundary value problem:

$$\begin{cases} -\Delta u = cu + F(u, \nabla u), & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

where

$$S := \nabla : H_0^1(\Omega) \to L^2(\Omega; \mathbb{R}^n).$$

In what follows, we present basic results from the abstract linear variational theory (see [8]). Let $A : D(A) \subset H \to H$ be a linear operator with $D(A)$ dense in H, *positively defined,* i.e., *symmetric* in the sense that $\langle Au, v\rangle_H = \langle u, Av\rangle_H$ for every $u, v \in D(A)$ and with $\langle Au, u\rangle_H \geq \gamma^2 \|u\|_H^2$ for every $u \in D(A)$ and some constant $\gamma^2 > 0$. The linear subspace $D(A)$ is endowed with the inner product

$$\langle u, v\rangle_{H_A} := \langle Au, v\rangle_H$$

and the *energetic norm* $\|u\|_{H_A} := \sqrt{\langle u, u\rangle_{H_A}}$. The completion of $(D(A),$ $\langle ., .\rangle_{H_A})$ is called the *energetic space* of A and is denoted by H_A. Since A is positively and densely defined, we have

$$\|u\|_H \leq \frac{1}{\gamma} \|u\|_{H_A} \quad \text{for all } u \in H_A. \tag{1.3}$$

From the Riesz representation theorem, it follows that for each $f \in H$, there exists a unique $u_f \in H_A$ with $\langle u_f, v\rangle_{H_A} = \langle f, v\rangle_H$ for every $v \in H_A$. We denote u_f by $A^{-1}f$ and we called it the *weak* (or *generalized*) *solution* of the equation $Au = f$. Thus

$$A^{-1} : H \to H_A \subset H, \quad \left\langle A^{-1}f, v\right\rangle_{H_A} = \langle f, v\rangle_H \quad \text{for } f \in H,\ v \in H_A. \tag{1.4}$$

Clearly, the linear operator A^{-1} from H to H is symmetric. From (1.3) and (1.4) we have

$$\left\langle A^{-1}f, f\right\rangle_H = \left\|A^{-1}f\right\|^2_{H_A} \geq \gamma^2 \left\|A^{-1}f\right\|^2_H.$$

Hence A^{-1} is positively defined. In addition, A^{-1} is completely continuous if the embedding of H_A into H is completely continuous. In what follows we assume that the embedding of H_A into H is completely continuous. Then, from the general theory of eigenvalues and eigenvectors of linear, positively defined and completely continuous operators, see [2] and [14], we know that: all eigenvalues of A^{-1} are positive; the set of eigenvalues of A^{-1} is nonempty and at most countable; zero is the only possible cluster point of it; there exists an orthonormal set (ϕ_k) of eigenvectors of A^{-1} which is at most countable and it is complete in the image of A^{-1}, i.e.,

$$A^{-1}u = \sum \left\langle A^{-1}u, \phi_k\right\rangle_H \phi_k \quad \text{for all } u \in H. \tag{1.5}$$

Assume that $D(A)$ is infinite dimensional. Then the image of A^{-1} is infinite dimensional and so there exists a sequence $(\mu_k)_{k\geq 1}$ of eigenvalues of A^{-1} and correspondingly, an orthonormal (in H) sequence $(\phi_k)_{k\geq 1}$ of eigenvectors. Let $\lambda_k := \frac{1}{\mu_k}$. Then $0 =: \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq$

... , $\lambda_k \to \infty$ as $k \to \infty$, and from $A^{-1}\phi_k = \mu_k \phi_k$, we have

$$\langle \phi_k, v \rangle_{H_A} = \lambda_k \langle \phi_k, v \rangle_H \quad \text{for all } v \in H_A,$$

i.e., $A\phi_k = \lambda_k \phi_k$ in the weak sense. Hence λ_k and ϕ_k $(k \geq 1)$ are the *eigenvalues* and *eigenvectors* of A with $\|\phi_k\|_H = 1$. From (1.5) we have that $(\phi_k)_{k\geq 1}$ is an Hilbert base in H and the sequence $(\lambda_k^{-1/2}\phi_k)_{k\geq 1}$ is an Hilbert base in H_A.

Now if c is a real number with $c \neq \lambda_k$ for all $k \geq 1$, we denote

$$\mu_c := \max \left\{ |c - \lambda_k|^{-1} : k = 1, 2, ... \right\}. \tag{1.6}$$

We shall use the following lemma [13] (see also [12]).

Lemma 1.1 *If $c \neq \lambda_k$ for all $k \geq 1$, then for each $v \in H$, there exists a unique weak solution $u \in H_A$ to the equation*

$$Lu := Au - cu = v$$

denoted by $L^{-1}v$, and the following eigenvector expansion holds

$$L^{-1}v = \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \phi_k \tag{1.7}$$

where the series converges in H_A. In addition,

$$\left\| L^{-1}v \right\|_H \leq \mu_c \|v\|_H. \tag{1.8}$$

We shall look for weak solutions to problem (1.1), i.e. an element $u \in H_A$ with

$$\langle u, w \rangle_{H_A} = c \langle u, w \rangle_H + \langle F(u, Su), w \rangle_H, \tag{1.9}$$

for all $w \in H_A$. If we look a priori for a solution u to (1.1) of the form $u = L^{-1}v$, with $v \in H$, then we have to solve a fixed point problem in H :

$$T(v) = v,$$

where

$$T : H \to H, \quad T(v) = F(L^{-1}v, SL^{-1}v). \tag{1.10}$$

2 Nonresonance Existence Theory

Our first result is an existence, uniqueness and approximation theorem.

Theorem 2.1 *We suppose that*

$$\lambda_j < c < \lambda_{j+1} \text{ for some } j \in \mathbb{N},\ j \geq 1,\ \text{ or } 0 \leq c < \lambda_1. \tag{2.1}$$

Also, we assume that

$$\|F(u_1, v_1) - F(u_2, v_2)\|_H \leq a \|u_1 - u_2\|_H + b \|v_1 - v_2\|_Y , \tag{2.2}$$

for all $u_1, u_2 \in H$, $v_1, v_2 \in Y$, where a, b are two nonnegative constants such that

$$a\mu_c + b\sqrt{\mu_c(1 + c\mu_c)} < 1, \tag{2.3}$$

with μ_c given by (1.6)*. Then, problem* (1.1) *has a unique solution $u \in H_A$. In addition,*

$$T^n(v_0) \to v \text{ in } H \text{ as } n \to \infty$$

for all $v_0 \in H$, where $u = L^{-1}v$.

Proof. We show that T is a contraction on H. For this, let $v_1, v_2 \in H$. Then, from (2.2) and (1.2) we have

$$\begin{aligned} \|T(v_1) - T(v_2)\|_H &\leq a \left\|L^{-1}(v_1 - v_2)\right\|_H + b \left\|SL^{-1}(v_1 - v_2)\right\|_Y \\ &\leq a \left\|L^{-1}(v_1 - v_2)\right\|_H + b \|S\| \left\|L^{-1}(v_1 - v_2)\right\|_{H_A} \\ &\leq a \left\|L^{-1}(v_1 - v_2)\right\|_H + b \left\|L^{-1}(v_1 - v_2)\right\|_{H_A} . \end{aligned} \tag{2.4}$$

From (1.8) we have

$$\left\|L^{-1}(v_1 - v_2)\right\|_H \leq \mu_c \|v_1 - v_2\|_H . \tag{2.5}$$

Next, note that by (1.7) and (1.6), for all $v \in H$ we have

$$
\begin{aligned}
\left\langle v, L^{-1}v \right\rangle_H &= \left\langle v, \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \left\langle v, \phi_k \right\rangle_H \phi_k \right\rangle_H \\
&\leq \mu_c \sum_{k=1}^{\infty} \left\langle v, \phi_k \right\rangle_H^2 \\
&= \mu_c \left\| v \right\|_H^2 .
\end{aligned}
\tag{2.6}
$$

Now, using (1.9), (2.5) and (2.6) we deduce

$$
\begin{aligned}
\left\| L^{-1}(v_1 - v_2) \right\|_{H_A}^2 &= c \left\| L^{-1}(v_1 - v_2) \right\|_H^2 + \left\langle v_1 - v_2, L^{-1}(v_1 - v_2) \right\rangle_H \\
&\leq c\mu_c^2 \left\| v_1 - v_2 \right\|_H^2 + \mu_c \left\| v_1 - v_2 \right\|_H^2 \\
&\leq \mu_c(1 + c\mu_c) \left\| v_1 - v_2 \right\|_H^2 .
\end{aligned}
$$

Consequently, (2.4) implies

$$
\left\| T(v_1) - T(v_2) \right\|_H \leq \left(a\mu_c + b\sqrt{\mu_c(1 + c\mu_c)} \right) \left\| v_1 - v_2 \right\|_H .
$$

This together with (2.3) shows that T is a contraction. The conclusion follows now from Banach's contraction principle. ■

The next theorem is an existence result derived from Schauder's fixed point principle, assuming that nonlinearity F has a growth at most linear.

Theorem 2.2 *We suppose that* (2.1) *holds, F is continuous and satisfies the growth condition*

$$
\left\| F(u, v) \right\|_H \leq a \left\| u \right\|_H + b \left\| v \right\|_Y + h,
\tag{2.7}
$$

for all $u \in H$, $v \in Y$, where $h \in \mathbb{R}_+$ and a, b are as in (2.3). *We also assume that there exists a Hilbert space Z, with $Z \subset Y$ and such that*

(2.8) *the embedding of Z into Y is completely continuous.*

In addition, we assume that

$$
L^{-1}(H) \subset S^{-1}(Z)
\tag{2.9}
$$

and that

(2.10) $\quad S$ *is continuous from* $H_A \cap S^{-1}(Z)$ *to* Z.

Then, problem (1.1) *has at least one solution* $u \in H_A \cap S^{-1}(Z)$.

Proof. First we prove that T given by (1.10) is completely continuous. It is obvious that T is a continuous operator. Let $M \subset H$ a bounded subset. As in the proof of Theorem 2.2 from [9],

(2.11) $\quad L^{-1}(M)$ is a relatively compact subset of H

since the embedding of H_A into H is completely continuous. Next, since L^{-1} is continuous from H to H_A we have that $L^{-1}(M)$ is bounded in H_A. Also, by (2.9) we have that $L^{-1}(M) \subset S^{-1}(Z)$. Thus, $L^{-1}(M)$ is bounded in $H_A \cap S^{-1}(Z)$. Now, from (2.10) we deduce that $SL^{-1}(M)$ is a bounded subset of Z. Consequently,

(2.12) $\quad SL^{-1}(M)$ is relatively compact in Y,

since the embedding of Z into Y is completely continuous. Finally, by (2.11) and (2.12) $F\left(L^{-1}(M), SL^{-1}(M)\right)$ is relatively compact in H, since F is a continuous operator from $H \times Y$ into H.

Next, we prove that T is a self-map of a sufficiently large closed ball of H. Similar estimates to those in the proof of Theorem 2.2 show that

$$\begin{aligned} \|T(v)\|_H &\leq a\left\|L^{-1}v\right\|_H + b\left\|SL^{-1}v\right\|_Y + h \\ &\leq a\left\|L^{-1}v\right\|_H + b\|S\|\left\|L^{-1}v\right\|_{H_A} + h \\ &\leq a\left\|L^{-1}v\right\|_H + b\left\|L^{-1}v\right\|_{H_A} + h \\ &\leq \left(a\mu_c + b\sqrt{\mu_c(1+c\mu_c)}\right)\|v\|_H + h. \end{aligned}$$

Finally, condition (2.3) assures that we can choose

$$R \geq \frac{h}{1-\left(a\mu_c + b\sqrt{\mu_c(1+c\mu_c)}\right)} > 0$$

such that T is a self-map of the ball $\overline{B}(0,R)$ of H. The conclusion follows now from Schauder's fixed point theorem. ■

Remark 2.1 Assumption (2.9) represents an abstract regularity condition.

The next result is based on the Leray-Schauder principle.

Theorem 2.3 *We suppose that F is continuous, bounded and has the decomposition*

$$F(u,v) = G(u,v) + F_0(u,v) + F_1(u,v).$$

Also, we assume that

$$0 \le c \le \beta < \lambda_1\,, \tag{2.13}$$

$$\|F_0(u,v)\|_H \le a\,\|u\|_H + b\,\|v\|_Y + h_0\,, \tag{2.14}$$

$$\|F_1(u,v)\|_H \le a_1\,\|u\|_H + b_1\,\|v\|_Y + h_1\,, \tag{2.15}$$

$$\langle F_1(u,v), u\rangle \le 0, \tag{2.16}$$

$$\langle G(u,v), u\rangle_H \le (\beta - c)\,\|u\|_H^2\,, \tag{2.17}$$

for all $u \in H$, $v \in Y$,where $a, a_1, b, b_1, h_0, h_1, \beta \in \mathbb{R}_+$. We also assume that there exists a Hilbert space Z, with $Z \subset Y$ such that the embedding of Z into Y is completely continuous. In addition, we assume that (2.9), (2.10) *hold and that*

$$a/\lambda_1 + b/\sqrt{\lambda_1} < 1 - \beta/\lambda_1. \tag{2.18}$$

Then, problem (1.1) *has at least one solution $u \in H_A \cap S^{-1}(Z)$.*

Proof. We look for a fixed point $w \in H$ of T. As above, T is a completely continuous operator. We show that the set of all solutions to

$$w = \lambda T(w), \tag{2.19}$$

when $\lambda \in (0,1)$, is bounded in H. Let $w \in H$ be any solution of (2.19). Let $u = L^{-1}w$. It is clear that u solves

$$\begin{cases} Au - cu = \lambda F(u, Su) \\ u \in H_A \end{cases} \tag{2.20}$$

in the weak sense. Since u is a weak solution of (2.20), we have

$$\|u\|_{H_A}^2 = \langle cu + \lambda F(u, Su), u \rangle_H . \tag{2.21}$$

From (2.17) we deduce

$$\begin{aligned} \langle cu + \lambda G(u, Su), u \rangle_H &= c \langle u, u \rangle_H + \lambda \langle G(u, Su), u \rangle_H \\ &\le c \|u\|_H^2 + (\beta - c) \|u\|_H^2 = \beta \|u\|_H^2 . \end{aligned} \tag{2.22}$$

We define

$$R(u) := \|u\|_{H_A}^2 - \beta \|u\|_H^2 . \tag{2.23}$$

Using (2.21), (2.22) and (2.16) we obtain

$$\begin{aligned} R(u) &= \langle cu + \lambda G(u, Su), u \rangle_H + \lambda \langle F_0(u, Su), u \rangle_H \\ &+\lambda \langle F_1(u, Su), u \rangle - \beta \|u\|_H^2 \le |\langle F_0(u, Su), u \rangle_H| . \end{aligned} \tag{2.24}$$

On the other hand, if we denote

$$c_k = \langle u, \phi_k \rangle_H = \langle u, \phi_k \rangle_{H_A} / \lambda_k ,$$

we see that

$$\begin{aligned} R(u) &= \sum_{k=1}^{\infty} (\lambda_k - \beta) c_k^2 \ge \sum_{k=1}^{\infty} \lambda_k (1 - \beta/\lambda_1) c_k^2 \\ &\ge (1 - \beta/\lambda_1) \|u\|_{H_A}^2 . \end{aligned} \tag{2.25}$$

Recall that

$$\lambda_1 = \inf \left\{ \|u\|_{H_A}^2 / \|u\|_H^2 \, ; \; u \in H_A \setminus \{0\} \right\}.$$

Now using (2.25), (2.24), (2.14), (1.2) and the fact that A is a positively defined operator, we obtain

$$\begin{aligned}
(1-\beta/\lambda_1)\|u\|_{H_A}^2 &\le |\langle F_0(u,Su),u\rangle_H| \\
&\le \|F_0(u,Su)\|_H \|u\|_H \\
&\le (a\|u\|_H + b\|Su\|_Y + h_0)\|u\|_H \\
&\le a\|u\|_H^2 + b\|u\|_{H_A}\|u\|_H + h_0\|u\|_H \\
&\le \tfrac{a}{\lambda_1}\|u\|_{H_A}^2 + \tfrac{b}{\sqrt{\lambda_1}}\|u\|_{H_A}^2 + C\|u\|_{H_A}
\end{aligned}$$

for $C = \frac{h_0}{\sqrt{\lambda_1}} > 0$. Hence (2.18) guarantees that there is a constant $r > 0$, independent of λ, such that $\|u\|_{H_A} \le r$. Finally, a bound for $\|w\|_H$ can be immediately derived from $u = L^{-1}w$ and (2.19). The conclusion now follows from the Leray-Schauder principle. ■

When $G = F_1 = 0$, Theorem 2.3 reduces to Theorem 2.2 for $j = 1$, since $\beta = c < \lambda_1$ and (2.18) is equivalent to (2.3). Indeed, first note that (2.13) implies $\mu_c = \frac{1}{\lambda_1 - c}$. Afterwards, (2.18) is equivalent to

$$a + b\sqrt{\lambda_1} < \lambda_1 - \beta. \tag{2.26}$$

Condition (2.3) is equivalent to

$$\frac{a}{\lambda_1 - c} + b\sqrt{\frac{1}{\lambda_1 - c}\left(1 + \frac{c}{\lambda_1 - c}\right)} < 1,$$

or, equivalently

$$a + b\sqrt{\lambda_1} < \lambda_1 - c. \tag{2.27}$$

Now (2.26) and (2.27) are the same since $\beta = c$.

The next result does not contain any regularity condition.

Theorem 2.4 *We suppose that F is continuous, bounded and has the* *decomposition*

$$F(u,v) = G(u) + F_0(u,v) + F_1(u).$$

We also assume that

$$\|F_0(u_1,v_1) - F_0(u_2,v_2)\|_H \le a\,\|u_1 - u_2\|_H + b\,\|v_1 - v_2\|_Y\,,$$

$$\|F_1(u)\|_H \le a_1\,\|u\|_H + h,$$
$$\langle F_1(u), u\rangle_H \le 0,$$

$$\langle G(u), u\rangle_H \le (\beta - c)\,\|u\|_H^2\,,$$

for all $u, u_1, u_2 \in H,\ v_1, v_2 \in Y$, *where* $a, a_1, b, h \in \mathbb{R}_+$. *In addition, we assume that* (2.13) *and* (2.18) *hold. Then, problem* (1.1) *has at least one solution* $u \in H_A$.

Proof. First note that, as above, (2.18) implies (2.3).

Let $T = T_0 + T_1$, where

$$T_0(v) = F_0\left(L^{-1}v, SL^{-1}v\right)$$

and

$$T_1(v) = (G \circ L^{-1})(v) + \left(F_1 \circ L^{-1}\right)(v),$$

for all $v \in H$. T_1 is a completely continuous operator. Next, as in the proof of Theorem 2.1, one can prove that T_0 is a contraction, since (2.18) implies (2.3). Consequently, T is a set-contraction. Next, the a priori bound of solutions is obtained by essentially the same reasoning as in the proof of Theorem 2.3 from [9]. ■

Notice that when $G = F_1 = 0$, Theorem 2.4 reduces to Theorem 2.1 for $0 \le \beta = c < \lambda_1$.

3 Application to semilinear elliptic equations

In this section we apply the general theory developed in the previous section to the weak solvability of the following boundary value problem:

$$\begin{cases} -\Delta u = cu + F(u, \nabla u), & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

Here, F is a general continuous operator

$$F : L^2(\Omega) \times L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega),$$

the constant c is not an eigenvalue of the operator $-\Delta$, while the general operator S from Section 2 is

$$S := \nabla : H_0^1(\Omega) \to L^2(\Omega; \mathbb{R}^n). \tag{3.2}$$

In this case $A = -\Delta,\ D(A) = C_0^2\left(\overline{\Omega}\right),\ H = L^2(\Omega),\ H_A = H_0^1(\Omega),$ and $Y = L^2(\Omega; \mathbb{R}^n)$.

By a solution of problem (3.1) we understand an element $u \in H_A$ with

$$\langle u, w \rangle_{H_0^1} = c \langle u, w \rangle_{L^2} + \langle F(u, \nabla u), w \rangle_{L^2},$$

for all $w \in H_0^1(\Omega)$.

First we show that (1.2) from the general theory holds. Indeed, the operator $S = \nabla$ defined by (3.2) is obviously a linear and continuous operator. Moreover, we have that

$$\|Su\|_{L^2(\Omega;\mathbb{R}^n)} = \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)} = \|u\|_{H_0^1}.$$

Thus, $\|S\| = 1$, so (1.2) is satisfied.

We now rewrite problem (3.1) as a fixed point problem. We define the operator $L : C_0^2\left(\overline{\Omega}\right) \to L^2(\Omega),\ Lu = Au - cu$. The operator L has all the properties described in Lemma 1.1. Let $L^{-1} : L^2(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega)$ be the inverse of L in the sense of Lemma 1.1. If we look a priori for a solution u to (3.1) of the form $u = L^{-1}v$, with $v \in L^2(\Omega)$, then we have to solve a fixed point problem in $L^2(\Omega)$: $T(v) = v$, where

$$T : L^2(\Omega) \to L^2(\Omega), \quad T(v) = F(L^{-1}v, \nabla L^{-1}v). \tag{3.3}$$

Theorem 3.1 *We suppose that*

$$\lambda_j < c < \lambda_{j+1} \text{ for some } j \in \mathbb{N},\ j \geq 1, \text{ or } 0 \leq c < \lambda_1. \tag{3.4}$$

Also, we assume that

$$\|F(u_1, v_1) - F(u_2, v_2)\|_{L^2} \leq a \|u_1 - u_2\|_{L^2} + b \|v_1 - v_2\|_{L^2(\Omega,\mathbb{R}^n)}, \tag{3.5}$$

for all $u_1, u_2 \in L^2(\Omega)$, $v_1, v_2 \in L^2(\Omega, \mathbb{R}^n)$, *where* a, b *are two nonnegative constants such that*

$$a\mu_c + b\sqrt{\mu_c(1 + c\mu_c)} < 1, \tag{3.6}$$

with μ_c *given by* (1.6). *Then, problem* (3.1) *has a unique solution* $u \in H_0^1(\Omega)$. *In addition,* $T^n(v_0) \to v$ *in* $L^2(\Omega)$ *as* $n \to \infty$ *for all* $v_0 \in L^2(\Omega)$, *where* $u = L^{-1}v$.

Proof. The proof is based on Theorem 2.1. Conditions (3.4), (3.5) and (3.6) imply (2.1), (2.2) and (2.3), respectively. The conclusion now follows by applying Theorem 2.1. ■

Theorem 3.2 *Suppose that* Ω *is* C^2, (3.4) *holds,* F *is continuous and satisfies the growth condition*

$$\|F(u, v)\|_{L^2} \leq a \|u\|_{L^2} + b \|v\|_{L^2(\Omega,\mathbb{R}^n)} + h, \tag{3.7}$$

for all $u \in L^2(\Omega)$, $v \in L^2(\Omega, \mathbb{R}^n)$, *where* $h \in \mathbb{R}_+$ *and* a, b *are as in* (3.6). *Then problem* (3.1) *has at least one solution* $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. The proof is based on Theorem 2.2. Let the space Z from the general theory be $Z := H^1(\Omega, \mathbb{R}^n)$. In this case we have $S^{-1}(Z) = H^2(\Omega)$. Since Ω is C^2, by the Rellich-Kondrachov theorem, the embedding of $H^1(\Omega, \mathbb{R}^n)$ into $L^2(\Omega, \mathbb{R}^n)$ is completely continuous, so condition (2.8) holds. Furthermore, it is well known that if Ω is C^2, then

$(-\Delta)^{-1}\left(L^2(\Omega)\right) \subset H^2(\Omega)$ and the linear map $(-\Delta)^{-1}$ is also bounded from $L^2(\Omega)$ into $H^2(\Omega)$. Consequently, $L^{-1}\left(L^2(\Omega)\right) \subset H^2(\Omega)$, so (2.9) holds as well. The operator $S = \nabla$ is a continuous operator from $H_0^1(\Omega) \cap H^2(\Omega)$ to $H^1(\Omega, \mathbb{R}^n)$, so (2.10) is true. Finally, (3.7) implies (2.7) and the conclusion follows now by applying Theorem 2.2. ■

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