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NONLINEAR SCHRÖDINGER EQUATIONS VIA FIXED POINT PRINCIPLES

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Abstract. This paper deals with weak solvability of the Cauchy-Dirichlet problem for the perturbed time-dependent Schrödinger equation. We use the operator approach based on fixed point theorems and properties of norm estimation and compactness of the solution operator associated to the nonhomogeneous linear Schrödinger equation. Also applied previously by the second author to nonlinear heat and wave equations, our operator method provides a unified way for treating different types of nonlinear boundary value problems.

Keywords. Nonlinear Schrödinger equation, weak solution, nonlinear operator, fixed point.

AMS (MOS) subject classification: 35Q55, 47J35.

1 Introduction

This paper deals with weak solvability of the Cauchy-Dirichlet problem for the perturbed Schrödinger equation:

$$\begin{cases} u_t - i\Delta u = \Phi(u) & \text{in } \ \Omega \times (0,T) \\ u(x,0) = g\,(x) & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbf{R}^n$ is a bounded domain and Φ is a general nonlinear operator which, in particular, can be a superposition operator, a delay operator, or an integral operator. Specific Schrödinger equations arise as models from several areas of physics. The problem is a classical one (see [2-6] and [11]) and our goal here is to make more precise the operator approach based on abstract results from nonlinear functional analysis. More exactly, we shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the nonhomogeneous linear Schrödinger equation and we shall use them in order to apply the Banach and Schauder theorems to the fixed point problem equivalent to problem (1.1). A similar programme has been applied to discuss nonlinear perturbations of the heat and wave equations in [8-10].

Compared to [8-10], here all spaces consist of complex-valued functions. Thus $L^2(\Omega)$ is the space of all complex-valued measurable functions u with $\int_\Omega |u(x)|^2 dx < \infty$ endowed with inner product and norm

$$(u,v)_{L^2} = \int_\Omega u(x)\overline{v(x)}dx, \quad |u|_{L^2} = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Also the Sobolev space of complex-valued functions $H_0^1(\Omega)$ is endowed with inner product and norm

$$(u,v)_{H_0^1} = \int_\Omega \left(\sum_{k=1}^n \frac{\partial u}{\partial x_k}\frac{\partial \overline{v}}{\partial x_k}\right)dx, \quad |u|_{H_0^1} = (u,u)_{H_0^1}^{\frac{1}{2}}.$$

As usual by $H^{-1}(\Omega)$ we denote the dual of $H_0^1(\Omega)$, that is the space of all linear continuous complex-valued functionals on $H_0^1(\Omega)$. The duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is defined as follows: for $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, (f,u) stands for the valued of f at $\overline{u}$; in particular, if $f \in L^1_{loc}(\Omega)$, then $(f,u) = \int_\Omega f\overline{u}dx$, and if $f \in L^2(\Omega)$, then $(f,u) = (f,u)_{L^2}$. Recall that $-\Delta$ is an isometry between spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ (see, e.g., [1] and [8]).

Throughout this paper by λ_k and ϕ_k $(k = 1, 2, ...)$ we mean the eigenvalues and eigenfunctions of $-\Delta$. Thus

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } \Omega \\ \phi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

If we assume that $|\phi_k|_{L^2} = 1$, then the systems $(\phi_k)_{k\geq 1}$, $\left(\frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{k\geq 1}$ are orthonormal and complete in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively.

2 The nonhomogeneous Schrödinger equation in $H^{-1}(\Omega)$

First we need the following lemma, the version for complex-valued functions of a result from [8], which is a generalization to $H^{-1}(\Omega)$ of Parseval's relation and of the completeness property of eigenfunctions ϕ_k. We include its proof for the sake of completeness.

Lemma 2.1 **(a)** *For any* $u \in H^{-1}(\Omega)$, *one has*

$$u = \sum_{k=1}^\infty (u,\phi_k)\phi_k \quad \left(in \;\; H^{-1}(\Omega)\right) \tag{2.1}$$

and

$$\sum_{k=1}^\infty \frac{1}{\lambda_k}|(u,\phi_k)|^2 = |u|^2_{H^{-1}}. \tag{2.2}$$

(b) *If* $u \in L^2(0,T;H^{-1}(\Omega))$, *then*

$$u = \sum_{k=1}^{\infty} (u, \phi_k)\, \phi_k \qquad \left(in \;\; L^2(0,T;H^{-1}(\Omega))\right).$$

Proof. (a) We use the fact that $-\Delta$ is an isometry between spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Thus, if $u \in H^{-1}(\Omega)$, then $(-\Delta)^{-1} u \in H_0^1(\Omega)$ and equality (2.1) is equivalent to

$$(-\Delta)^{-1} u = \sum_{k=1}^{\infty} (u, \phi_k)\, (-\Delta)^{-1} \phi_k \quad \left(\text{in} \;\; H_0^1(\Omega)\right).$$

Since $(-\Delta)^{-1} \phi_k = \frac{1}{\lambda_k} \phi_k$ and $(u, \phi_k) = \left((-\Delta)^{-1} u, \phi_k\right)_{H_0^1}$, the last equality can be rewritten as

$$(-\Delta)^{-1} u = \sum_{k=1}^{\infty} \left((-\Delta)^{-1} u,\; \frac{1}{\sqrt{\lambda_k}} \phi_k\right)_{H_0^1} \frac{1}{\sqrt{\lambda_k}} \phi_k \quad \left(\text{in} \;\; H_0^1(\Omega)\right).$$

But this equality is true since the system $\left(\frac{1}{\sqrt{\lambda_k}} \phi_k\right)$ is complete in $H_0^1(\Omega)$.

Equality (2.2) is equivalent to Parseval's equality in $H_0^1(\Omega)$ for the function $(-\Delta)^{-1} u$.

(b) According to (2.2) one has

$$|u(t)|_{H^{-1}}^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left|(u(t), \phi_k)\right|^2 \qquad \text{for a.e.} \;\; t \in [0,T].$$

Thus the problem reduces to the convergence in $L^1(0,T)$ of the sequence of partial sums. This happens by the Lebesgue dominated convergence theorem since the partial sums are dominated by the function $t \mapsto |u(.)|_{H^{-1}}^2$ belonging to $L^1(0,T)$. ∎

Remark 2.1 If the eigenfunctions ϕ_k are normalized in $L^2(\Omega)$, then:

(i) one has

$$|\phi_k|_{L^2} = 1, \quad |\phi_k|_{H_0^1} = \sqrt{\lambda_k}, \quad |\phi_k|_{H^{-1}} = \frac{1}{\sqrt{\lambda_k}};$$

(ii) the systems (ϕ_k), $\left(\frac{\phi_k}{\sqrt{\lambda_k}}\right)$, $\left(\sqrt{\lambda_k}\phi_k\right)$ are othonormal and complete in $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively;

(iii) due to the embeddings $L^2(\Omega) \subset H_0^1(\Omega) \subset H^{-1}(\Omega)$ and to the completeness of the system (ϕ_k) in all the three spaces, the Fourier series of a function $u \in L^2(\Omega)$ with respect to the three systems from (ii), in $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively, that is

$$\sum (u, \phi_k)_{L^2} \phi_k, \quad \sum \left(u, \frac{\phi_k}{\sqrt{\lambda_k}}\right)_{H_0^1} \frac{\phi_k}{\sqrt{\lambda_k}}, \quad \sum \left(u, \sqrt{\lambda_k}\phi_k\right)_{H^{-1}} \sqrt{\lambda_k}\phi_k,$$

are identical and can be written as $\sum (u, \phi_k) \phi_k$, where by (u, ϕ_k) we mean the action of u as an element of $H^{-1}(\Omega)$ over ϕ_k. Note that the scalar product in $H^{-1}(\Omega)$ is given by

$$(u, v)_{H^{-1}} := \left((-\Delta)^{-1} u, (-\Delta)^{-1} v\right)_{H_0^1}.$$

Consider the Cauchy-Dirichlet problem for the nonhomogeneous Schrödinger equation

$$\begin{cases} u_t - i\Delta u = f & \text{in } \Omega \times (0, T) \\ u(x, 0) = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{2.3}$$

We shall apply Fourier's method. For the first result, we shall work unitary in one of the spaces $H_0^1(\Omega)$, $L^2(\Omega)$, $H^{-1}(\Omega)$, which is denoted by V, endowed with the corresponding inner product $(.,.)_V$ and norm $|.|_V$ and we shall assume that $|\phi_k|_V = 1$ for every k. We already know that the system of eigenfunctions (ϕ_k) is complete in each of the three spaces. In what follows the notation $C^1([0, T]; \Delta V)$ is used to denote the space of all functions u with $(-\Delta)^{-1} u \in C^1([0, T]; V)$.

Theorem 2.1 *Assume that $g \in V$ and $f \in C([0, T]; V)$. Then there exists a unique function $u \in C([0, T]; V) \cap C^1([0, T]; \Delta V)$ with $u(0) = g$ and*

$$u'(t) - i\Delta u(t) = f(t) \quad \text{in } \Delta V \quad (t \in [0, T]). \tag{2.4}$$

In addition

$$|u(t)|_V^2 \leq 2\left(|g|_V^2 + t\int_0^t |f(s)|_V^2 \, ds\right), \qquad t \in [0, T]. \tag{2.5}$$

Proof. (a) We look for a solution in the form

$$u(t) = \sum_{k=1}^{\infty} u_k(t) \phi_k. \tag{2.6}$$

If we formally replace into (2.3) we obtain

$$u_k(t) = e^{-i\lambda_k t} g_k + \int_0^t e^{-i\lambda_k (t-s)} f_k(s) \, ds \tag{2.7}$$

where $f_k(t) = (f(t), \phi_k)_V$ and $g_k = (g, \phi_k)_V$.

(b) Series (2.6) defines a function $u \in C([0,T];V)$. Indeed its uniform convergence reduces to the uniform convergence of the series a real functions $\sum |u_k(t)|^2$. We have

$$\begin{aligned} |u_k(t)|^2 &\leq 2\left(|g_k|^2 + \left(\int_0^t |f_k(s)|\, ds\right)^2\right) \\ &\leq 2\left(|g_k|^2 + t\int_0^t |f_k(s)|^2\, ds\right). \end{aligned} \tag{2.8}$$

Thus the uniform convergence of $\sum |u_k(t)|^2$ is reduced to the convergence of the series of numbers $\sum |g_k|^2$ and $\sum \int_0^T |f_k(s)|^2\, ds$. According to Parseval's relation, the sum of the first series is $|g|_V^2$ while of the second one $\int_0^T |f(s)|_V^2\, ds$ since $f \in C([0,T];V)$.

(c) $u \in C^1([0,T];\Delta V)$. We have to prove that

$$v := (-\Delta)^{-1} u \in C^1([0,T];V).$$

One has

$$v(t) = \sum_{k=1}^{\infty} u_k(t)(-\Delta)^{-1}\phi_k = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} u_k(t)\phi_k.$$

Thus the problem is to show the uniform convergence in V of the series of derivatives

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} u_k'(t)\phi_k.$$

This reduces to the uniform convergence of the series $\sum \frac{1}{\lambda_k^2} |u_k'(t)|^2$. We have

$$u_k'(t) = -i\lambda_k e^{-i\lambda_k t} g_k + f_k(t) - i\lambda_k \int_0^t e^{-i\lambda_k(t-s)} f_k(s)\, ds.$$

Then

$$\begin{aligned} \frac{1}{\lambda_k^2} |u_k'(t)|^2 &\leq \frac{3}{\lambda_k^2}\left(\lambda_k^2 |g_k|^2 + |f_k(t)|^2 + t\lambda_k^2 \int_0^t |f_k(s)|^2\, ds\right) \\ &\leq 3\left(|g_k|^2 + \frac{1}{\lambda_1^2}|f_k(t)|^2 + t\int_0^t |f_k(s)|^2\, ds\right). \end{aligned}$$

So we are finished since $\sum |g_k|^2 = |g|_V^2$ and $\sum |f_k(t)|^2 = |f(t)|_V^2$ uniformly on $[0,T]$.

(d) Identity (2.4) follows if we pass to the limit in the corresponding identity for partial sums.

(e) Uniqueness. Assume $u_1,\ u_2$ are two such functions. Then $u := u_1 - u_2$ satisfies $u(0) = 0$ and $u'(t) - i\Delta u(t) = 0$ in ΔV, i.e., $(-\Delta)^{-1} u'(t) + iu(t) =$

0 in V. Denote $v := (-\Delta)^{-1} u$. Then $v(0) = 0$ and $v'(t) + iu(t) = 0$ in V. In particular, this gives

$$(v'(t), v(t))_V + i(u(t), v(t))_V = 0.$$

Observe that $(u(t), v(t))_V \in \mathbf{R}_+$ (verify for each space $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$), and that

$$Re\,(v'(t), v(t))_V = \frac{1}{2}\frac{d}{dt}|v(t)|_V^2.$$

Hence $\frac{d}{dt}|v(t)|_V^2 = 0$ on $[0,T]$. Thus $|v(t)|_V^2$ is constant and since $v(0) = 0$, we have $v(t) \equiv 0$, hence $u = 0$, that is $u_1 = u_2$.

(f) Relation (2.5) is an immediate consequence of (2.8). ∎

If f is less regular in x and more regular in t, then we have:

Theorem 2.2 *Assume that $g \in H_0^1(\Omega)$ and $f \in C^1([0,T]; H^{-1}(\Omega))$. Then there exists a unique function $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ with $u(0) = g$ and*

$$u'(t) - i\Delta u(t) = f(t) \quad \text{in} \quad H^{-1}(\Omega) \quad (t \in [0,T]).$$

In addition

$$|u(t)|_{H_0^1}^2 \le 4\left(|g|_{H_0^1}^2 + |f(t)|_{H^{-1}}^2 + |f(0)|_{H^{-1}}^2 + t\int_0^t |f'(s)|_{H^{-1}}^2\,ds\right)$$

and

$$|u'(t)|_{H^{-1}}^2 \le 3\left(|g|_{H_0^1}^2 + |f(0)|_{H^{-1}}^2 + t\int_0^t |f'(s)|_{H^{-1}}^2\,ds\right). \tag{2.9}$$

Proof. As above we look for a solution in the form (2.6), where this time we assume that $|\phi_k|_{L^2} = 1$. Hence in (2.7), $f_k(t) = (f(t), \phi_k)$ and $g_k = (g, \phi_k)$. First note that using derivative f_k', $u_k(t)$ can be written as

$$\begin{aligned} u_k(t) &= e^{-i\lambda_k t} g_k + \int_0^t e^{-i\lambda_k(t-s)} f_k(s)\,ds \\ &= e^{-i\lambda_k t} g_k - \frac{i}{\lambda_k}\int_0^t \left(e^{-i\lambda_k(t-s)}\right)' f_k(s)\,ds \\ &= e^{-i\lambda_k t} g_k - \frac{i}{\lambda_k} f_k(t) + \frac{i}{\lambda_k} e^{-i\lambda_k t} f_k(0) + \frac{i}{\lambda_k}\int_0^t e^{-i\lambda_k(t-s)} f_k'(s)\,ds. \end{aligned}$$

(a) $u \in C([0,T]; H_0^1(\Omega))$. This reduces to the uniform convergence of series $\sum \lambda_k |u_k(t)|^2$ which follows from the estimation

$$\begin{aligned} &\lambda_k |u_k(t)|^2 \\ \le\ & 4\lambda_k\left(|g_k|^2 + \frac{1}{\lambda_k^2}|f_k(t)|^2 + \frac{1}{\lambda_k^2}|f_k(0)|^2 + \frac{t}{\lambda_k^2}\int_0^t |f_k'(s)|^2\,ds\right) \\ \le\ & 4\left(\lambda_k|g_k|^2 + \frac{1}{\lambda_k}|f_k(t)|^2 + \frac{1}{\lambda_k}|f_k(0)|^2 + t\int_0^t \frac{1}{\lambda_k}|f_k'(s)|^2\,ds\right). \end{aligned}$$

(b) $u \in C^1\left([0,T]; H^{-1}(\Omega)\right)$. Follows from the estimation

$$\begin{aligned}
\frac{1}{\lambda_k}\left|u_k'(t)\right|^2 &= \frac{1}{\lambda_k}\left|-i\lambda_k e^{-i\lambda_k t} g_k + f_k(t) - i\lambda_k \int_0^t e^{-i\lambda_k(t-s)} f_k(s)\,ds\right|^2 \\
&= \frac{1}{\lambda_k}\left|-i\lambda_k e^{-i\lambda_k t} g_k + f_k(t) - \int_0^t \left(e^{-i\lambda_k(t-s)}\right)' f_k(s)\,ds\right|^2 \\
&= \frac{1}{\lambda_k}\left|-i\lambda_k e^{-i\lambda_k t} g_k + e^{-i\lambda_k t} f_k(0) + \int_0^t e^{-i\lambda_k(t-s)} f_k'(s)\,ds\right|^2 \\
&\leq 3\left(\lambda_k\left|g_k\right|^2 + \frac{1}{\lambda_k}\left|f_k(0)\right|^2 + t\int_0^t \frac{1}{\lambda_k}\left|f_k'(s)\right|^2 ds\right).
\end{aligned}$$

■

We shall associate to the Cauchy-Dirichlet problem with $g = 0$ for the Schrödinger equation, the following *solution operator:*

$$\begin{aligned}
S &: C([0,T]; H^{-1}(\Omega)) \to C([0,T]; H^{-1}(\Omega)), \\
f &\in C([0,T]; H^{-1}(\Omega)) \mapsto Sf \in C([0,T]; H^{-1}(\Omega)), \\
(Sf)(t) &= \sum_{k=1}^{\infty} u_k(t)\,\phi_k, \quad |\phi_k|_{L^2} = 1, \\
u_k(t) &= \int_0^t e^{-i\lambda_k(t-s)} f_k(s)\,ds, \quad f_k(t) = (f(t), \phi_k).
\end{aligned}$$

Hence Sf is the unique function u satisfying the conditions of Theorem 2.1, for $g = 0$.

3 Properties of the Schrödinger solution operator

3.1 Norm estimations

It follows from (2.5) and (2.9) that the solution operator S is a linear continuous self-mapping of

$$C\left([0,T]; H_0^1(\Omega)\right), \quad C\left([0,T]; L^2(\Omega)\right) \quad \text{and} \quad C\left([0,T]; H^{-1}(\Omega)\right).$$

Also

$$S\left(C^1\left([0,T]; H^{-1}(\Omega)\right)\right) \subset C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; H^{-1}(\Omega)\right).$$

Theorem 3.1 **(i)** *Let $f \in C([0,T]; V)$. Then for every $t \in [0,T]$,*

$$\left|(Sf)(t)\right|_V \leq \sqrt{2t}\,|f|_{L^2(0,t;V)}. \tag{3.1}$$

Here, as above, V is any of the spaces $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

(ii) *Let $f \in C^1\left([0,T]\,; H^{-1}(\Omega)\right)$. Then for every $t \in [0,T]$,*

$$|(Sf)(t)|^2_{H^1_0} \le 12\left(|f(0)|^2_{H^{-1}} + t\,|f'|^2_{L^2(0,t;H^{-1}(\Omega))}\right)$$

and

$$\left|(Sf)'(t)\right|^2_{H^{-1}} \le 3\left(|f(0)|^2_{H^{-1}} + t\,|f'|^2_{L^2(0,t;H^{-1}(\Omega))}\right).$$

Proof. (i) Simple consequence of (2.5).
(ii) Use (2.9) and

$$f(t) = f(0) + \int_0^t f'(s)\,ds,$$

whence

$$|f(t)|^2_{H^{-1}} \le 2\left(|f(0)|^2_{H^{-1}} + t\int_0^t |f'(s)|^2_{H^{-1}}\,ds\right).$$

■

3.2 Compactness

This subsection deals with the complete continuity of the solution operator S. We shall also use the following result (see [8] and [12]) :

Lemma 3.1 *Let X, B and Y be Banach spaces with the inclusion $X \subset B$ compact and $B \subset Y$ continuous. If a set F is bounded in $L^p(0,T;X)$ and relatively compact in $L^p(0,T;Y)$ where $1 \le p \le \infty$, then F is relatively compact in $L^p(0,T;B)$.*

Theorem 3.2 **(i)** *The solution operator is completely continuous from $C([0,T]\,;H^1_0(\Omega))$ to $C([0,T]\,;L^p(\Omega))$ for every $1 \le p < 2^*$.*

(ii) *The solution operator is completely continuous from $C^1([0,T]\,;H^{-1}(\Omega))$ to $C([0,T]\,;L^p(\Omega))$ for every $(2^*)' \le p < 2^*$.*

Proof. (i) Obviously, if M is a bounded subset of $C([0,T]\,;H^1_0(\Omega))$, then SM is bounded in $C([0,T]\,;H^1_0(\Omega))$, as shows Theorem 2.1. Consequently, SM is also bounded in $C([0,T]\,;L^p(\Omega))$, and for each t, $(SM)(t)$ is a bounded subset of $H^1_0(\Omega)$, thus relatively compact in $L^p(\Omega)$. It remains to prove that SM is equicontinuous in $C([0,T]\,;L^p(\Omega))$. To this end, let $f \in M$ and $u = Sf$. We have

$$|u_k(t) - u_k(t')|^2 = \left|\int_{t'}^t e^{-i\lambda_k(t-s)} f_k(s)\,ds\right|^2 \le |t-t'|\int_{t'}^t |f_k(s)|^2\,ds.$$

It follows that

$$|u(t) - u(t')|^2_{H^1_0} \le |t-t'|\int_0^T |f(s)|^2_{H^1_0}.$$

This together with $|u(t) - u(t')|_{L^p} \leq C\,|u(t) - u(t')|_{H_0^1}$ proves the equicontinuity of SM.

(ii) If M is bounded in $C^1([0,T]\,; H^{-1}(\Omega))$, then SM is bounded in $C([0,T]\,; H_0^1(\Omega))$ and in $C^1([0,T]\,; H^{-1}(\Omega))$. This implies that SM is relatively compact in $C\left([0,T]\,; H^{-1}(\Omega)\right)$. Then, from Lemma 3.1, we deduce that SM is relatively compact in $C([0,T]\,; L^p(\Omega))$. ■

4 Nonlinear Schrödinger equations

4.1 Applications of Banach's fixed point theorem

Our first existence and uniqueness result for the semilinear problem (1.1) is established by means of Banach's fixed point theorem.

Theorem 4.1 *Let $g \in L^2(\Omega)$ and $\Phi : C\left([0,T]; L^2(\Omega)\right) \to C\left([0,T]; L^2(\Omega)\right)$ be a map for which there exists a constant $a \in \mathbf{R}_+$ such that the following inequality holds for all $u,\ v \in C\left([0,T]; L^2(\Omega)\right)$:*

$$|\Phi(u)(t) - \Phi(v)(t)|_{L^2} \leq a\,|u(t) - v(t)|_{L^2} \quad \text{for every} \quad t \in [0,T]. \tag{4.1}$$

Then there exists a unique function

$$u \in C\left([0,T]\,; L^2(\Omega)\right) \cap C^1\left([0,T]; \Delta L^2(\Omega)\right)$$

such that

$$\begin{cases} u'(t) - i\Delta u(t) = \Phi(u)(t) \quad \text{in} \quad \Delta L^2(\Omega)\,, \quad \text{for every} \quad t \in [0,T]\,, \\ u(0) = g_0. \end{cases}$$

Proof. Let u_0 be the solution of problem (1.1) corresponding to $\Phi = 0$. We have to solve the fixed point problem

$$u = u_0 + (S \circ \Phi)(u)\,, \quad u \in C\left([0,T]\,; L^2(\Omega)\right).$$

The conclusion will follow from Banach's fixed point theorem once we have shown that the operator

$$A : C\left([0,T]\,; L^2(\Omega)\right) \to C\left([0,T]\,; L^2(\Omega)\right), \quad A(u) = u_0 + (S \circ \Phi)(u)$$

is a contraction with respect to a suitable norm on $C\left([0,T]\,; L^2(\Omega)\right)$. Let $u,\ v \in C\left([0,T]\,; L^2(\Omega)\right)$. We have

$$\begin{aligned} |A(u)(t) - A(v)(t)|_{L^2}^2 &= |S(\Phi(u) - \Phi(v))(t)|_{L^2}^2 \\ &\leq 2t\,|\Phi(u) - \Phi(v)|_{L^2(0,t;L^2(\Omega))}^2 . \end{aligned}$$

Let $\theta > Ta^2$ be a fixed number and consider the norm on $C\left([0,T];L^2(\Omega)\right)$,

$$\|u\| = \max_{t\in[0,T]}\left(|u(t)|_{L^2}\, e^{-\theta\, t}\right).$$

Furthermore,

$$\begin{aligned}
|\Phi(u)-\Phi(v)|^2_{L^2(0,t;L^2(\Omega))} &= \int_0^t |\Phi(u)(s)-\Phi(v)(s)|^2_{L^2}\, ds \\
&= a^2\int_0^t e^{2\theta s}e^{-2\theta s}\,|u(s)-v(s)|^2_{L^2}\, ds \\
&\le a^2\,\|u-v\|^2\int_0^t e^{2\theta s}ds \\
&= \frac{a^2}{2\theta}\,\|u-v\|^2\left(e^{2\theta t}-1\right) \\
&\le \frac{a^2}{2\theta}\,\|u-v\|^2\, e^{2\theta t}.
\end{aligned}$$

Hence

$$|A(u)(t)-A(v)(t)|^2_{L^2} \le \frac{ta^2}{\theta}\,\|u-v\|^2\, e^{2\theta t}.$$

Dividing by $e^{2\theta t}$ and taking the maximum over $[0,T]$ yields

$$\|A(u)-A(v)\| \le \frac{\sqrt{T}a}{\sqrt{\theta}}\,\|u-v\|.$$

Since $\sqrt{T}a/\sqrt{\theta} < 1$, the operator A is a contraction on $C\left([0,T];L^2(\Omega)\right)$ with respect to the norm $\|.\|$. ■

Example 4.1 Let $\Psi : L^2(\Omega) \to L^2(\Omega)$ be a map for which there exists a constant $a \in \mathbf{R}_+$ with

$$|\Psi(u)-\Psi(v)|_{L^2(\Omega)} \le a\,|u-v|_{L^2(\Omega)} \quad \textit{for all} \quad u,\ v\in L^2(\Omega). \tag{4.2}$$

Then the map $\Phi : C\left([0,T];L^2(\Omega)\right) \to C\left([0,T];L^2(\Omega)\right)$ given by

$$\Phi(u)(t) = \Psi(u(t)) \quad \left(u\in C\left([0,T];L^2(\Omega)\right),\ t\in[0,T]\right)$$

satisfies all the assumptions of Theorem 4.1.

Example 4.2 Let $\psi : \Omega\times\mathbf{C} \to \mathbf{C}$ be a continuous function such that $\psi(.,0)\in L^2(\Omega)$ and there is a constant $a\in\mathbf{R}_+$ with

$$|\psi(x,\tau_1)-\psi(x,\tau_2)| \le a\,|\tau_1-\tau_2|$$

for all $x\in\Omega$ and $\tau_1,\ \tau_2\in\mathbf{C}$. Then the operator $\Psi : L^2(\Omega)\to L^2(\Omega)$ defined by

$$\Psi(u) = \psi(.,u(.))$$

satisfies the condition from the previous example.

Indeed, we may write

$$\Psi(u) = \psi(.,0) + h(.,u(.))$$

where $h(x,\tau) = \psi(x,\tau) - \psi(x,0)$. Notice that $|h(x,\tau)| \leq a|\tau|$ (for all $x \in \Omega$ and $\tau \in \mathbf{C}$), so the superposition operator $h(.,u(.))$ (see [7]) maps $L^2(\Omega)$ into $L^2(\Omega)$ and $|h(.,u(.)) - h(.,v(.))|_{L^2} \leq a|u-v|_{L^2}$.

Denote

$$C_0^1\left([0,T];H^{-1}(\Omega)\right) = \left\{u \in C^1\left([0,T];H^{-1}(\Omega)\right): \ u(0) = 0\right\}.$$

Theorem 4.2 *Let $g \in H_0^1(\Omega)$ and let*

$$\Phi : C\left([0,T];H_0^1(\Omega)\right) \to C_0^1\left([0,T];H^{-1}(\Omega)\right)$$

be a map for which there exists a constant $a \in \mathbf{R}_+$ such that the following inequality holds for all $u, \ v \in C\left([0,T];H_0^1(\Omega)\right)$:

$$|\Phi(u)'(t) - \Phi(v)'(t)|_{H^{-1}} \leq a|u(t)-v(t)|_{H_0^1} \quad \text{for every} \ \ t \in [0,T].$$

Then there exists a unique function

$$u \in C\left([0,T];H_0^1(\Omega)\right) \cap C^1\left([0,T];H^{-1}(\Omega)\right)$$

satisfying

$$\begin{cases} u'(t) - i\Delta u(t) = \Phi(u)(t) \quad \text{in} \ \ H^{-1}(\Omega), \ \ \text{for every} \ \ t \in [0,T], \\ u(0) = g_0. \end{cases}$$

Proof. We have to solve the fixed point problem for the operator $A : C\left([0,T];H_0^1(\Omega)\right) \to C\left([0,T];H_0^1(\Omega)\right), \ A(u) = u_0 + (S \circ \Phi)(u)$. We have

$$\begin{aligned} |A(u)(t) - A(v)(t)|_{H_0^1}^2 &= |S(\Phi(u) - \Phi(v))(t)|_{H_0^1}^2 \\ &= 12t\int_0^t \left|\Phi(u)'(s) - \Phi(v)'(s)\right|_{H^{-1}}^2 ds \\ &\leq 12ta^2\int_0^t |u(s) - v(s)|_{H_0^1}^2 ds. \end{aligned}$$

As above, if we consider $\theta > 6Ta^2$ and norm $\|u\| = \max_{t\in[0,T]}\left(|u(t)|_{H_0^1} e^{-\theta t}\right)$ on $C\left([0,T];H_0^1(\Omega)\right)$, we obtain

$$\|A(u) - A(v)\| \leq a\sqrt{\frac{6T}{\theta}}\,\|u-v\|.$$

■

Example 4.3 Let $\Phi_0 : C\left([0,T]; H_0^1(\Omega)\right) \to C\left([0,T]; H^{-1}(\Omega)\right)$ be a map for which there is a constant $a \in R_+$ such that

$$|\Phi_0(u)(t) - \Phi_0(v)(t)|_{H^{-1}} \leq a\,|u(t) - v(t)|_{H_0^1}$$

for all $u,\ v \in C\left([0,T]; H_0^1(\Omega)\right)$ and $t \in [0,T]$. Then the map Φ given by

$$\Phi(u)(t) = \int_0^t \Phi_0(u)(s)\,ds$$

satisfies all the assumptions of Theorem 4.2.

4.2 Applications of Schauder's fixed point theorem

The next existence results are based on Schauder's fixed point theorem. The Lipschitz condition on the nonlinear term Φ in Theorem 4.1 is weakened to a condition of at most linear growth.

Theorem 4.3 *Let $g \in H_0^1(\Omega)$, $p \in [1, 2^*)$ and $\Phi : C([0,T]; L^p(\Omega)) \to C\left([0,T]; H_0^1(\Omega)\right)$ a continuous map for which there exists a constant $a \in \mathbf{R}_+$ such that the following inequality holds for every $u \in C([0,T]; L^p(\Omega))$:*

$$|\Phi(u)(t) - \Phi(0)(t)|_{H_0^1} \leq a\,|u(t)|_{L^p} \quad \textit{for every} \;\; t \in [0,T].$$

Then there exists at least one function

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; H^{-1}(\Omega)\right)$$

such that

$$\begin{cases} u'(t) - i\Delta u(t) = \Phi(u)(t) & \textit{in} \;\; H^{-1}(\Omega), \;\; \textit{for every} \;\; t \in [0,T], \\ u(0) = g_0. \end{cases}$$

Proof. We look for a fixed point of the operator

$$A : C([0,T]; L^p(\Omega)) \to C([0,T]; L^p(\Omega)), \quad N(u) = u_0 + (S \circ \Phi)(u).$$

Theorem 3.2 and the continuity and boundedness of the operator Φ guarantee the complete continuity of A. It remains to find a nonempty, bounded, closed and convex subset D of $C([0,T]; L^p(\Omega))$ with $A(D) \subset D$. Let $u \in C([0,T]; L^p(\Omega))$. As in the proof of Theorem 4.1, one obtains

$$\begin{aligned} |A(u)(t) - A(0)(t)|_{L^p} &\leq c\,|A(u)(t) - A(0)(t)|_{H_0^1} \\ &= c\,|S(\Phi(u) - \Phi(0))(t)|_{H_0^1} \\ &\leq 2ct\,|\Phi(u) - \Phi(0)|^2_{L^2\left(0,t;H_0^1(\Omega)\right)}. \end{aligned}$$

Since

$$|\Phi(u) - \Phi(0)|^2_{L^2\left(0,t;H_0^1(\Omega)\right)} = \int_0^t |\Phi(u)(s) - \Phi(0)(s)|^2_{H_0^1} ds \leq a^2 \int_0^t |u(s)|^2_{L^p} ds$$

we have

$$|A(u)(t) - A(0)(t)|^2_{L^p} \leq 2a^2 ct \int_0^t |u(s)|^2_{L^p} ds.$$

If in $C([0,T]; L^p(\Omega))$ we consider the norm $\|u\| = \max_{t\in[0,T]} \left(|u(t)|_{L^p} e^{-\theta t}\right)$, then we deduce

$$\|A(u) - A(0)\| \leq a\sqrt{\frac{cT}{\theta}} \|u\|.$$

Hence

$$\|A(u)\| \leq \|A(0)\| + a\sqrt{\frac{cT}{\theta}} \|u\|.$$

If we choose $\theta > a^2cT$, then we can find a large enough $R > 0$ such that $\|u\| \leq R$ implies $\|A(u)\| \leq R$. Thus, Schauder's fixed point theorem applies. ■

Similarly, we have

Theorem 4.4 *Let $g \in H_0^1(\Omega)$, $p \in [(2^*)', 2^*)$ and $\Phi : C([0,T]; L^p(\Omega)) \to C_0^1\left([0,T]; H^{-1}(\Omega)\right)$ be a continuous map for which there exists a constant $a \in \mathbf{R}_+$ such that the following inequality holds for every $u \in C([0,T]; L^p(\Omega))$:*

$$|\Phi(u)'(t) - \Phi(0)'(t)|_{H^{-1}} \leq a\,|u(t)|_{L^p} \quad \text{for every} \quad t \in [0,T].$$

Then there exists at least one function

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; H^{-1}(\Omega)\right)$$

satisfying

$$\begin{cases} u'(t) - i\Delta u(t) = \Phi(u)(t) & \text{in } H^{-1}(\Omega), \text{ for every } t \in [0,T], \\ u(0) = g_0. \end{cases}$$

Example 4.4 Let $\Phi_0 : C([0,T]; L^p(\Omega)) \to C\left([0,T]; H^{-1}(\Omega)\right)$ be a map for which there is a constant $a \in R_+$ such that

$$|\Phi_0(u)(t) - \Phi_0(v)(t)|_{H^{-1}} \leq a\,|u(t) - v(t)|_{L^p}$$

for all $u,\ v \in C([0,T]; L^p(\Omega))$ and $t \in [0,T]$. Then the map Φ given by

$$\Phi(u)(t) = \int_0^t \Phi_0(u)(s)\,ds$$

satisfies all the assumptions of Theorem 4.4.

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