USE OF PROBABILISTIC METHODS IN THE THEORY OF UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS

BY

D. D. STANCU

(Cluj)

In this paper one discusses a general probabilistic method for constructing positive linear operators useful in the theory of uniform approximation of continuous functions. One derives by this method the known operators of Bernstein, Mirakyan, Baskakov, Weierstrass, a variant of Meyer-König and Zeller operator, an operator of Feller and also an operator which was recently introduced and studied in detail by the author of the present paper. For the operators of discrete type it is further indicated a way for representing them in terms of the factorial moments of the distribution involved and of the finite differences of the corresponding function. By means of the modulus of continuity of the function used and of the standard deviation of the random variable involved, one gives evaluations of the orders of approximation by the considered operators.

§ 1. PROBABILISTIC METHODS FOR CONSTRUCTING POSITIVE LINEAR OPERATORS

1. Consider a sequence of one-dimensional random variables $\{Y_n\}$, and let $F_n(y; x)$ be the distribution function of Y_n, where y is any real number and x is a one-dimensional real parameter, varying in a finite or infinite interval I, representing the expectation of this distribution.

Let $f = f(y)$ be a real-valued function bounded on the real axis, such that the mean value of the random variable $f(Y_n)$ exists ($n = 1, 2, 3, \ldots$). As is well known, this mean value can be expressed by the improper Stieltjes integral of $f(y)$ with respect to $F_n(y; x)$:

$$E[f(Y_n)] = P_n(f; x) = \int_{-\infty}^{\infty} f(y)\, \mathrm{d}_y F_n(y; x). \tag{1.1}$$

REV. ROUM. MATH. PURES ET APPL., TOME XIV, Nº 5, p. 673-691, BUCAREST, 1969

If Y_n is a discrete random variable then the distribution $F_n(y; x) = P[Y_n \leq y; x]$ is a step function with a finite or countable infinite number of jump points $\{a_k\}$ on the real axis, at which jumps $\{p_n(a_k)\}$ occur. The jump of $F_n(y; x)$ at a_k is given by

$$p_n(a_k) = p_n(a_k;\ x) = P[Y_n = a_k;\ x] = F_n(a_k;\ x) - F_n(a_k - 0;\ x).$$

The function $p_n(y; x)$ represents the discrete probability function, which is characterized by the fact that it is positive if y coincides with a point of the sequence $\{a_k\}$, is zero if we have no jumps at y, and we have

$$\sum_k p_n(a_k;\ x) = 1.$$

In this discrete case we can write

$$F_n(y;\ x) = \sum_k p_n(a_k;\ x), \tag{1.2}$$

the sum extending over all those a_k which do not exceed y, and thus the operator (1.1) can be expressed as follows

$$P_n(f;\ x) = \sum_k f(a_k)\, p_n(a_k;\ x). \tag{1.3}$$

If the random variable Y_n has a continuous probability density function $\rho_n(y; x)$, then we have

$$F_n(y;\ x) = \int_{-\infty}^{y} \rho_n(u;\ x)\, \mathrm{d}u,$$

and we obtain the following expression for the operator (1.1):

$$P_n(f;\ x) = \int_{-\infty}^{\infty} f(y)\, \rho_n(y;\ x)\, \mathrm{d}y. \tag{1.4}$$

It is evident that the operator $P_n(f; x)$ defined by the formula (1.1), or in particular by the formula (1.3) or (1.4), is a *positive linear operator.*

2. We shall now make use of an important method for constructing concrete operators of this type, useful in the theory of uniform approximation of continuous functions.

Let us assume that Y_n represents the arithmetic mean of the first n terms of a sequence $\{X_m\}$ of random variables, that is

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \quad (n = 1, 2, \ldots). \tag{1.5}$$

If for any natural number n the random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed, then denoting by $\varphi(t)$ the common characteristic function of these random variables, the characteristic function of Y_n will be (see, e.g., [9] or [21])

$$\psi_n(t) = \left[\varphi\left(\frac{t}{n}\right)\right]^n. \tag{1.6}$$

We now proceed to give some examples, assuming that, in the case of the first six of these, $\{X_m\}$ is a sequence of independent random variables identically distributed.

(a) Consider X_j $(j = 1, 2, \dots)$ has a zero-one distribution, i.e.,

$$P(X_j = 0) = 1 - x, \ P(X_j = 1) = x \quad (j = 1, 2, \dots),$$

where $0 \leqq x \leqq 1$.

In this case the characteristic function of X_j $(j = 1, 2, \dots)$ is: $\varphi(t) = 1 - x + xe^{it}$. Consequently, the characteristic function of Y_n will be

$$\psi_n(t) = \left(1 - x + xe^{\frac{it}{n}}\right)^n,$$

which corresponds to the binomial distribution

$$b(k;\, n,\, x) = \binom{n}{k} x^k (1 - x)^{n-k}. \tag*{1.7)}$$

Hence in this case we can write

$$F_n(y;\, x) = \begin{cases} 0 & \text{if} \quad y < 0 \\ \sum\limits_{0 \leqq k \leqq ny} b(k;\, n,\, x) & \text{if} \quad 0 \leqq y \leqq 1 \\ 1 & \text{if} \quad y \geqq 1. \end{cases}$$

According to (1.3) we obtain the operator

$$P_n(f;\, x) = B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \tag{1.8}$$

since in this case the jump points are $a_k = \frac{k}{n}$ $(k = 0, 1, \dots, n)$ and for the jumps we have the values $p_n(a_k;\, x) = b(k;\, n,\, x)$ $(k = 0, 1, \dots, n)$.

Thus, we arrived at the famous Bernstein polynomials, introduced in 1912 by Bernstein [2] using a rather analogous probabilistic method. We mention that Goncharov [10], Favard [6], Lorentz [14], Parzen [22], Feller [7] and Lamperti [13] also indicated quite similar probabilistic ways for obtaining the Bernstein polynomials.

(b) Let $X_j (j = 1, 2, \dots)$ have the Poisson distribution

$$P(X_j = k) = p(k;\, x) = e^{-x} \frac{x^k}{k!} \quad (k = 0, 1, 2, \dots),$$

where x is a positive parameter.

In this case the characteristic functions of $X_j (j = 1, 2, \dots)$ and Y_n are respectively

$$\varphi(t) = e^{x(e^{it} - 1)}, \quad \psi_n(t) = e^{nx(e^{\frac{it}{n}} - 1)}.$$

Hence Y_n has a Poisson distribution with the parameter nx. Therefore, now we have

$$F_n(y\,;\,x) = \begin{cases} 0 & \text{if} \quad y < 0 \\ \sum\limits_{0 \leqq k \leqq ny} p\,(k\,;\,x) & \text{if} \quad y \geqq 0, \end{cases}$$

and formula (1.3) leads us to the following operator of Mirakyan [18]:

$$P_n(f\,;\,x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k\,!}\,\mathrm{e}^{-nx} f\left(\frac{k}{n}\right). \tag{1.9}$$

Favard [6], Szasz [28], Feller [7] and Butzer and Berens [4] discussed briefly the probabilistic methods for obtaining this operator.

(c) Suppose X_j $(j = 1, 2, \ldots)$ have a geometric distribution

$$P\,(X_j = k) = x\,(1 - x)^k \quad (k = 0,\ 1,\ 2, \ldots),$$

where $0 \leqq x \leqq 1$. The corresponding characteristic function is

$$\varphi\,(t) = x \sum_{k=0}^{\infty} ((1 - x)\,\mathrm{e}^{it})^k = \frac{x}{1 - (1 - x)\,\mathrm{e}^{it}}.$$

It follows that the characteristic function of Y_n is

$$\psi_n(t) = \left(\frac{x}{1 - (1 - x)\,\mathrm{e}^{\frac{it}{n}}}\right)^n,$$

which is the characteristic function of $Y = \dfrac{X}{n}$, where X has the negative binomial (Pascal) distribution:

$$\begin{aligned} P\,(X = k) = p\,(k\,;\,n,\,x) &= \binom{n + k - 1}{k} x^n (1 - x)^k \\ &= (-1)^k \binom{-\,n}{k} x^n (1 - x)^k \qquad (k = 0,\ 1,\ 2, \ldots). \end{aligned} \tag{1.10}$$

The corresponding distribution function is

$$F_n(y\,;\,x) = \begin{cases} 0 & \text{if} \quad y < 0 \\ \sum\limits_{0 \leqq k \leqq ny} p\,(k\,;\,n,\,x) & \text{if} \quad y \geqq 0. \end{cases}$$

By virtue of (1.3) in this case we have the operator

$$P_n(f\,;\,x) = \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^n (1 - x)^k f\left(\frac{k}{n}\right). \tag{1.11}$$

In [17] Meyer-König and Zeller investigated the following operator

$$M_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1-x)^n f\left(\frac{k}{n+k}\right), \tag{1.12}$$

which differs slightly from the preceding one (it can be obtained from (1.11) if we replace $1-x$ by x and the basic points $\dfrac{k}{n+k}$ by $\left.\dfrac{k}{n}\right)$.

It should be noticed that Cheney and Sharma [5] gave an important generalization of the operator (1.12).

(d) If we replace x by $\dfrac{1}{1+x}$ in (1.11) then we get the operator of Baskakov [1]

$$P_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \tag{1.13}$$

which, as a matter of fact, corresponds to the following negative binomial distribution [8], a variant of the distribution (1.10),

$$q(k;n,x) = p$$

$$p\left(k;n,\frac{1}{1+x}\right) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \qquad (k=0,1,2,\ldots). \tag{1.14}$$

It should be noticed that the operators (1.9), (1.11) and (1.13) may be defined also for non-integral n.

(e) Assume X_j $(j=1,2,\ldots)$ are continuous, having the normal distribution

$$\rho(y;x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}},$$

with the mean value x and the standard deviation σ $(\sigma > 0)$.

Now we have

$$\varphi(t) = e^{ixt - \frac{\sigma^2 t^2}{2}}, \qquad \psi_n(t) = e^{ixt - \frac{\sigma^2 t^2}{2n}}$$

and it is obvious that Y_n has a normal distribution with the mean value x and the standard deviation $\dfrac{\sigma}{\sqrt{n}}$.

In this case (1.4) yields the operator

$$P_n^{(\sigma)}(f;x) = \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{n(y-x)^2}{2\sigma^2}}\, dy. \tag{1.15}$$

In the special case $\sigma = 2^{-\frac{1}{2}}$ it reduces to the classic Weierstrass [29] operator

$$W_n(f;x) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-n(y-x)^2}\,\mathrm{d}y.$$

(f) Let X_j $(j = 1, 2, \ldots)$ have the exponential distribution

$$\rho\,(y\,;x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ \dfrac{1}{x}\,\mathrm{e}^{-\frac{y}{x}} & \text{if} \quad x > 0. \end{cases}$$

Now we obtain

$$\varphi\,(t) = \frac{1}{1 - ixt}, \quad \psi_n(t) = \frac{1}{\left(1 - i\dfrac{x}{n}\,t\right)^n}.$$

Since for the gamma distribution

$$\rho\,(y\,;x, n) = \begin{cases} 0 & \text{if} \quad y < 0 \\ \dfrac{y^{n-1}}{\Gamma(n)}\,\mathrm{e}^{-\frac{y}{x}}\,\dfrac{1}{x^n} & \text{if} \quad y \geqq 0, \end{cases}$$

where $x > 0$, the characteristic function is (see, e.g., [12])

$$\psi\,(t) = \frac{1}{(1 - ixt)^n},$$

it follows that Y_n has a gamma distribution, with the continuous probability density $\rho_n(y\,;x) = \rho\left(y\,;\dfrac{x}{n}, n\right)$.

Consequently, in this case we obtain the operator

$$P_n(f;x) = \frac{1}{\Gamma(n)}\left(\frac{n}{x}\right)^n\int_0^{\infty} f(y)\,y^{n-1}\,\mathrm{e}^{-\frac{ny}{x}}\,\mathrm{d}y, \tag{1.16}$$

encountered in 1966 by Feller [7].

In his doctoral thesis Müller [19] studied in detail the following variant of operators, obtained by starting from the gamma distribution

$$G_n(f;x) = \frac{x^{n+1}}{n\,!}\int_0^{\infty} \mathrm{e}^{-xy}\,y^n f\left(\frac{n+1}{y}\right)\mathrm{d}y.$$

In connection with this operator references should also be made to the papers [15], [20].

(g) Now we shall consider an example of random variables which are not independent.

Consider the Markov-Pólya urn scheme ([16], [24]). An urn contains a white and b black balls; a ball is drawn at random and if it is white (black) it is replaced and moreover, c white (black) balls are added. This procedure is repeated n times. Let X_j be one or zero according as the jth trial results in white or black. It is known that the probability that the total number of white balls : $X_1 + X_2 + \ldots + X_n$ be equal with $k\,(0 \leqq k \leqq n)$ is given by (see, e.g., [7], [9]) :

$$p(k;\ n, a, b, c) = \binom{n}{k} \frac{a\,(a+c) \ldots (a+\overline{k-1}c)\; b\,(b+c) \ldots (b+\overline{n-k-1}\,c)}{N\,(N+c) \ldots (N+\overline{n-1}c)},$$

where $N = a + b$.

In this case the variables $X_j\ (j = 1, 2, \ldots)$ are symmetrically dependent (see [7]).

Let $\dfrac{a}{N} = x$, $\dfrac{c}{N} = \alpha$. Since $\dfrac{b}{N} = 1 - x$, we see that the probability of

$$Y_n = \frac{X_1 + X_2 + \ldots X_n}{n} = \frac{k}{n}$$

is

(1.17)
$$w_{n,k}\,(x;\ \alpha) =$$

$$\binom{n}{x} \frac{x\,(x+\alpha) \ldots (x+\overline{k-1}\alpha)\,(1-x)\,(1-x+\alpha) \ldots (1-x+\overline{n-k-1}\alpha)}{(1+\alpha)\,(1+2\alpha) \ldots (1+\overline{n-1}\alpha)}$$

and the corresponding distribution function will be

$$F_n^{[\alpha]}\,(y\,;\,x) = \begin{cases} 0 & \text{if } y < 0 \\ \displaystyle\sum_{0 \leqq k \leqq ny} w_{n,k}\,(x\,;\,\alpha) & \text{if } 0 \leqq y \leqq 1 \\ 1 & \text{if } y \geqq 1. \end{cases}$$

In this case we obtain the operator

$$P_n^{[\alpha]}\,(f\,;\,x) = \sum_{k=0}^{n} w_{n,k}\,(x\,;\,\alpha) f\left(\frac{k}{n}\right), \tag{1.18}$$

introduced and studied in detail by Stancu [26] in the case when the parameter α is a non-negative parameter.

We should remark that in the special case $\alpha = 0$ this operator reduces, obviously, to the classic Bernstein operator (1.8), and in the case $\alpha = -\dfrac{1}{n}$ it becomes (surprisingly!) the Lagrange interpolation polynominal corresponding to the function f and the equally spaced nodes $\dfrac{k}{n}$ $(k = 0, 1, \ldots, n)$, namely

$$P_n^{\left[-\frac{1}{n}\right]}(f\,;\,x) = L_n\left(f\,;\,0,\ \frac{1}{n},\ \ldots,\frac{n}{n}\ ;\,x\right) = \sum_{k=0}^{n} l_{n,k}\,(x)\, f\left(\frac{k}{n}\right),$$

where

$$l_{n,k}(x)=(-1)^{n-k}\frac{n!}{k!\,(n-k)!}\,x\left(x-\frac{1}{n}\right)\dots\left(x-\frac{k-1}{n}\right)\left(x-\right.$$

$$\left.-\frac{k+1}{n}\right)\dots\left(x-\frac{n}{n}\right).$$

As we have shown in [26], the operator (1.9) can be obtained from (1.18) as a limiting case.

We notice that the operator (1.13) of Baskakov can also be obtained as the limit of our operator (1.18). Indeed, let us assume that α and x depend on n in such a way that for n large the values of α and x are small, whereas $n\alpha = t$, $nx = mt$ are of moderate magnitude. We have

$$\binom{n}{k}x\,(x+\alpha)\,(x+2\alpha)\dots(k+\overline{k-1}\,\alpha)=$$

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)t^k m\,(m+1)\dots(m+k-1)\to\binom{m+k-1}{k}t^k.$$

On the other hand

$$\frac{(1-x)\,(1-x+\alpha)\dots(1-x+\overline{n-k-1}\,\alpha)}{(1+\alpha)\,(1+2\alpha)\dots(1+\overline{n-1}\,\alpha)}=$$

$$\prod_{j=0}^{n-k-1}\left(1-\frac{mt}{n}+j\,\frac{t}{n}\right)\Big/\prod_{i=0}^{n-1}\left(1+i\,\frac{t}{n}\right)=\left[1\Big/\prod_{j=n-k}^{n=1}\left(1-\frac{mt}{n}+\right.\right.$$

$$\left.\left.+\,j\,\frac{t}{n}\right)\right]\left[\prod_{j=0}^{n-1}\left(1-\frac{mt}{n}+j\,\frac{t}{n}\right)\Big/\prod_{i=0}^{n-1}\left(1+i\,\frac{t}{n}\right)\right]\longrightarrow\frac{1}{(1+t)^k}\cdot\frac{1}{(1+t)^m},$$

since the following result is true (see problem 57, part II, chap. I of [23]):

$$\lim_{n\to\infty}\frac{a}{b}\cdot\frac{a+d}{b+d}\cdot\frac{a+2d}{b+2d}\dots\frac{a+(n-1)\,d}{b+(n-1)\,d}=(1+\delta)^{\frac{\alpha-\beta}{\delta}},$$

where

$$a=1+\frac{\alpha}{n},\ b=1+\frac{\beta}{n},\ d=\frac{\delta}{n},$$

α, β and δ being fixed and $\delta > 0$.

Hence

$$w_{n,k}(x;\ \alpha)\to\binom{m+k-1}{k}\frac{t^k}{(1+t)^{m+k}}.$$

If we take into account that for $x=\dfrac{k}{n}$ we find from $nx = mt$ that $t=\dfrac{k}{m}$, we arrive at the operator $P_m(f;\,t)$ defined at (1.13).

§ 2. REPRESENTATIONS BY FINITE DIFFERENCES

3. We shall now discuss an approach to the problem of representing by finite differences of some of the linear positive operators considered in the preceding section.

To deal with this problem it will be convenient to consider the Newton interpolation polynomial of a function $f(t)$ with respect to the nodes $\frac{k}{n}$ $(k = 0, 1, \ldots, n)$:

$$N(f; t) = N\left(f; 0, \frac{1}{n}, \ldots, \frac{n}{n}; t\right)$$
$$= f(0) + \sum_{j=1}^{n} \frac{(nt)^{[j]}}{j!} \Delta^{j}_{\frac{1}{n}} f(0),$$

where: $(nt)^{[j]} = nt(nt-1)\ldots(nt-j+1)$, while $\Delta^{j}_{\frac{1}{n}} f(0)$ represents the finite difference of f with the step $\frac{1}{n}$ and the starting point zero.

By making the change of variable $nt = y$ we obtain

$$N\left(f; \frac{y}{n}\right) = f(0) + \sum_{j=1}^{n} \frac{y^{[j]}}{j!} \Delta^{j}_{\frac{1}{n}} f(0). \tag{2.1}$$

This polynomial enjoys the interpolating properties

$$N\left(f; \frac{k}{n}\right) = f\left(\frac{k}{n}\right) (k = 0, 1, \ldots, n). \tag{2.2}$$

In accordance with (2.1) we can derive the following formula for the mean value of the random variable $N\left(f; \frac{y}{n}\right)$, where $y = Y$ has the distribution function $F_n(y; x)$:

$$\int_{-\infty}^{\infty} N\left(f; \frac{y}{n}\right) d_y F_n(y; x) = f(0) + \sum_{j=1}^{n} \frac{m_{[j]}}{j!} \Delta^{j}_{\frac{1}{n}} f(0), \tag{2.3}$$

while

$$m_{[j]} = \int_{-\infty}^{\infty} y^{[j]} \, d_y F_n(y; x) \quad (j = 1, 2, \ldots, n) \tag{2.4}$$

represent the corresponding factorial moments.

It is known that the factorial moment-generating function is defined by

$$g(t) = E(t^{y}) = \int_{-\infty}^{\infty} t^{y} \, d_y F_n(y; x). \tag{2.5}$$

Obviously, we have $m_{[j]} = g^{[j]}(1)$ $(j = 1, 2, \ldots, n)$.

If the random variable Y is discrete and assumes the values k $(k = 0, 1, \ldots, n)$ with the probabilities $p_{n,k}$ $(k = 0, 1, \ldots, n)$, then the equation (2.3), by virtue of (2.2), reduces to

$$\sum_{k=0}^{n} N\left(f; \frac{k}{n}\right) p_{n,k} =$$

$$\sum_{k=0}^{n} p_{n,k} f\left(\frac{k}{n}\right) = f(0) + \sum_{j=1}^{n} \frac{m_{[j]}}{j!} \Delta_{\frac{1}{n}}^{j} f(0), \tag{2.6}$$

where

$$m_{[j]} = \sum_{k=0}^{n} k^{[j]} p_{n,k}. \quad = m_{[j]}(x)$$

In this case the factorial moment-generating function will be

$$g(t) = \sum_{k=0}^{n} t^{k} p_{n,k}. \quad = g(t;x) \tag{2.7}$$

Examples.

(a) Let us consider first that Y has the binomial distribution (1.7) Since

$$g(t) = \sum_{k=0}^{n} t^{k} b(k; n, x) = \sum_{k=0}^{n} \binom{n}{k} (tx)^{k} (1 - x)^{n-k} = (1 - x + tx)^{n},$$

we have

$$g^{(j)}(t) = n^{[j]} x^{j} (1 - x + tx)^{n-j}, \quad m_{[j]} = n^{[j]} x^{j}$$

and (2.6) leads us to the following representation for the Bernstein polynomial (1.8):

$$B_n(f; x) = \sum_{j=0}^{n} \binom{n}{j} x^{j} \Delta_{\frac{1}{n}}^{j} f(0). \tag{2.8}$$

As we know this formula is due to Grüss [11].

If we assume that x depends on n in such a way that for $n \to \infty$ we have $nx \to z > 0$, then $\binom{n}{j} x^{j} \to \frac{1}{j!} z^{j}$ as $n \to \infty$, and we get from (2.8) the following representation for the Mirakyan operator (1.9):

$$P_n(f; z) = \sum_{j=0}^{\infty} \frac{(nz)^{j}}{j!} \Delta_{\frac{1}{n}}^{j} f(0) \tag{2.9}$$

(b) Now consider the Markov-Pólya distribution: $P(Y = k) = w_{n,k}(x; \alpha)$, this probability being defined by (1.17).

Let us find the corresponding factorial moments. We have

$$g(t) = \sum_{k=0}^{n} t^{k} w_{n,k}(x; \alpha), \quad m_{[j]} = g^{(j)}(1) = \sum_{k=j}^{n} k^{[j]} w_{n,k}(x; \alpha).$$

Let

$$N_n = (1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\,\alpha).$$

We can write successively

$$m_{[j]} = \frac{1}{N_n}\sum_{k=j}^{n} k^{[j]}\,\frac{n^{[k]}}{k!}\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)$$

$$= \frac{1}{N_n}\sum_{k=j}^{n} \frac{n^{[k]}}{(k-j)!}\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)$$

$$= \frac{1}{N_n}\sum_{s=0}^{n-j} \frac{n^{[j+s]}}{s!}\prod_{\nu=0}^{j+s-1}(x+\nu\alpha)\prod_{\mu=0}^{n-j-s-1}(1-x+\mu\alpha),$$

by setting $k-j=s$, s being the new summation variable.

Since

$$\prod_{\nu=0}^{j+s-1}(x+\nu\alpha) = \prod_{\nu'=0}^{j-1}(x+\nu'\alpha)\prod_{\nu''=j}^{j+s-1}(x+\nu''\alpha),$$

we can write further

$$m_{[j]} = \frac{n!}{N_n}\prod_{\nu'=0}^{j-1}(x+\nu'\alpha)\sum_{s=0}^{n-j}\left[\frac{1}{s!}\prod_{\nu''=j}^{j+s-1}(x+ \nu''\alpha)\right]\left[\frac{1}{(n-j-s)!}\prod_{\mu=0}^{n-j-s-1}(1-x+\mu\alpha)\right]=$$

$$= n^{[j]}\frac{x(x+\alpha)\ldots(x+\overline{j-1}\,\alpha)}{(1+\alpha)\ldots(1+\overline{n-1}\,\alpha)}\sum_{s=0}^{n-j}\binom{n-j}{s}(x+j\alpha)^{[s/-\alpha]}(1-x)^{[(n-j-s)/-\alpha]},$$

where

$$(x+j\alpha)^{[s/-\alpha]} = \prod_{\nu''=j}^{j+s-1}(x+\nu''\alpha) = (x+j\alpha)(x+\overline{j+1}\,\alpha)\ldots$$

$$\ldots(x+\overline{j+s-1}\,\alpha),$$

$$(1-x)^{[(n-j-s)/-\alpha]} = \prod_{\mu=0}^{n-j-s-1}(1-x+\mu\alpha) = (1-x)(1-x+\alpha)\ldots$$

$$\ldots(1-x+\overline{n-j-s-1}\,\alpha).$$

Making use of the Vandermonde formula

$$(a+b)^{[m/-\alpha]} = \sum_{s=0}^{m}\binom{m}{s}a^{[s/-\alpha]}b^{[(m-s)/-\alpha]},$$

we find that

$$\sum_{s=0}^{n-j}\binom{n-j}{s}(x+j\alpha)^{[s/-\alpha]}(1-x)^{[(n-j-s)/-\alpha]} = (1+j\alpha)^{[(n-j)/-\alpha]} =$$

$$= (1+j\alpha)(1+\overline{j+1}\alpha)\dots(1+\overline{n-1}\alpha).$$

Thus, we have

$$m_{[j]} = n^{[j]}\,\frac{x(x+\alpha)\dots(x+\overline{j-1}\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{j-1}\alpha)}.$$

Consequently, formula (2.6) leads us to the following representation for our operators (1.18):

$$P_n^{[\alpha]}(f;x) = f(0) + \sum_{j=1}^{n}\binom{n}{j}\frac{x(x+\alpha)\dots(x+\overline{j-1}\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{j-1}\alpha)}\,\Delta^{j}_{\frac{1}{n}}f(0), \tag{2.10}$$

which in the case $\alpha > 0$ has been established by us in the paper [26].

In the limit case which led us from the operator (1.18) to the operator (1.13), obviously we have

$$\binom{n}{j}\frac{x(x+\alpha)\dots(x+\overline{j-1}\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{j-1}\alpha)}\,\Delta^{j}_{\frac{1}{n}}f(0) \to \binom{m+j-1}{j}t^{j}\,\Delta^{j}_{\frac{1}{m}}f(0).$$

Thus the operator $P_m(f;t)$ defined at (1.13) can be represented under the form

$$P_m(f;t) = f(0) + \sum_{j=1}^{\infty}\binom{m+j-1}{j}t^{j}\,\Delta^{j}_{\frac{1}{m}}f(0). \tag{2.11}$$

If we replace t by $(1-x)/x$ $(0 < x < 1)$ then we obtain an expression in terms of finite differences of f for the operator (1.11).

REMARK. Since in the special case $f(x) = n^r x^r$, where r is a non-negative integer, we have $\Delta^{j}_{\frac{1}{n}}f(0) = \Delta^{j}x^{r}|_{x=0} = \Delta^{j}0^{r}$, the formulas (2.8), (2.9), (2.10) and (2.11) permit us to express immediately the ordinary moments of the distributions: binomial-$b(k;\ n, p)$, Poisson-$p\ (k;\lambda)$ Markov-Pólya-$P\ (k;\ n,\ a,\ b,\ c)$ and the negative binomial-$q(k;\ n,\ p)$ and $p\,(k;\ n,\ p)$, respectively by the following formulas, which use the differences of zero,

$$\nu_r(n;p) = \sum_{j=1}^{r}\binom{n}{j}p^{j}\,\Delta^{j}0^{r},$$

$$\nu_r(\lambda) = \sum_{j=1}^{r}\frac{\lambda^{j}}{j\,!}\,\Delta^{j}0^{r},$$

$$\nu_r(n\,;a,b,c)=\sum_{j=1}^{r}\binom{n}{j}\frac{a\,(a+c)\ldots(a+\overline{j-1}c)}{(a+b)\,(a+b+c)\ldots(a+b+\overline{j-1}c)}\,\Delta^j0^r,$$

$$\nu_r(n\,;p)=\sum_{j=1}^{r}\binom{n+j-1}{j}p^j\,\Delta^j0^r,$$

$$\nu_r(n,p)=\sum_{j=1}^{r}\binom{n+j-1}{j}\left(\frac{q}{p}\right)^j\Delta^j0^r,$$

where $0<p<1$, $p+q=1$ and in each case : $\nu_0=1$. The first of this is due to Bohlmann [3]; the second one is a limit case of it. The third one was obtained recently by us [27] in the case $c>0$.

§ 3. APPLICATIONS TO THE APPROXIMATION OF CONTINUOUS FUNCTIONS

4. In this section we shall deal with the problem of approximation of a bounded uniformly continuous real-valued function $f=f(x)$ on the real axis by the operators $P_n(f;\,x)$.

In order to evaluate the corresponding orders of approximation it is convenient to make use of the modulus of continuity of f, which is defined by

$$\omega(f;\,\delta)=\sup|f(x'')-f(x')| \tag{3.1}$$

for x' and x'' real numbers such that $|x''-x'|\leqslant\delta$, δ being a positive number.

We will need the following known property of the modulus of continuity

$$\omega(f;\,\lambda\delta)\leqq(\lambda+1)\,\omega(f;\,\delta)\quad(\lambda>0). \tag{3.2}$$

Now let us consider the variance of the random variable Y_n, i.e. the mean value of the function $f(y)=(y-x)^2$:

$$\mathrm{Var}\,[Y_n]=\int_{-\infty}^{\infty}(y-x)^2\,d_yF_n(y,x)=\sigma_n^2(x),$$

where x, the mean value of $F_n(y,x)$, belongs to the finite or infinite interval I and $\sigma_n(x)$ is the standard deviation of Y_n.

It is easy to prove the following result.

THEOREM 3.1. *If f is a bounded uniformly continuous real-valued function on the real axis then*

$$|f(x)-P_n(f;\,x)|\leqq(1+\beta_n\sqrt{n})\,\omega\left(f;\,\frac{1}{\sqrt{n}}\right) \tag{3.3}$$

where $\beta_n=\sup\sigma_n(x)$ for $x\in I$.

Proof. Since

$$\int_{-\infty}^{\infty}d_yF_n(y\,;x)=1$$

we can write

$$f(x) - P_n(f; x) = \int_{-\infty}^{\infty} [f(x) - f(y)]\, d_y F_n(y; x).$$

Consequently

$$|f(x) - P_n(f; x)| \leqq \int_{-\infty}^{\infty} |f(x) - f(y)|\, d_y F_n(y; x).$$

But, by (3.1) and (3.2) we have

$$|f(x) - f(y)| \leqq \omega(f; \delta^{-1}|x-y|\delta) \leqq (1 + \delta^{-1}|x-y|)\, \omega(f; \delta) \quad (\delta > 0)$$

Therefore we obtain

$$(3.4) \qquad |f(x) - P_n(f; x)| \leqq (1 + \delta^{-1} w_n(x))\, \omega(f; \delta),$$

where

$$w_n(x) = \int_{-\infty}^{\infty} |x - y|\, d_y F_n(y; x).$$

Since by the Schwarz inequality we have $w_n(x) \leqq \sigma_n(x)$ for $x \in I$ we can write

$$(3.5) \quad |f(x) - P_n(f; x)| \leqq (1 + \delta^{-1}\sigma_n(x))\, \omega(f; \delta) \leqq (1 + \delta^{-1}\beta_n)\, \omega(f; \delta).$$

By taking $\delta = n^{-\frac{1}{2}}$ we obtain just the inequality (3.3).

Remark. Let $b_n \geqq \beta_n$ and $\gamma > 0$; by taking $\delta = \gamma b_n$ in (3.5) we get

$$(3.6) \qquad |f(x) - P_n(f; x)| \leqq \left(1 + \frac{1}{\gamma}\right) \omega(f; \gamma b_n).$$

From Theorem 3.1 there follows immediately

Corollary 3.1. *If f is uniformly continuous on I and $\beta_n \to 0$ as $n \to \infty$ i.e.,*

$$\lim_{n\to\infty} \max_{I} \sigma_n(x) = 0,$$

then the sequence of operators $\{P_n(f; x)\}$ is uniformly convergent towards f on I.

Illustrations. 1°. For the Bernstein operator (1.8) we have

$$I = [0, 1], \quad \sigma_n(x) = \sqrt{\frac{x(1-x)}{n}}, \quad \beta_n = \frac{1}{2\sqrt{n}}, \ \lambda = 2.$$

Hence (3.3) leads us to the inequality

$$|f(x) - B_n(f; x)| \leqq \frac{3}{2}\, \omega\left(f; \frac{1}{\sqrt{n}}\right),$$

due to Popoviciu [25].

2°. In the case of operator of Mirakyan (1.9) we take $I = [0, a]$, where $0 < a < \infty$, and we have

$$\sigma_n(x) = \sqrt{\frac{x}{n}}, \quad \beta_n = \sqrt{\frac{a}{n}}, \ \lambda = \frac{1}{\sqrt{a}}$$

Consequently we obtain from (3.3) the inequality

$$|f(x) - P_n(f;x)| \leq (1+\sqrt{a})\,\omega\left(f; \frac{1}{\sqrt{n}}\right),$$

which was found recently by Müller [19].

3°. For the operator (1.11) we have

$$\bar{\sigma}_n(x) = P_n\left(\left(\frac{1-x}{x} - y\right)^2;\ x\right) = \frac{1}{x}\sqrt{\frac{1-x}{n}}.$$

Taking $I = [a, 1]$, where $0 < a < 1$ we find

$$\left|f\left(\frac{1-x}{x}\right) - P_n(f;\ x)\right| \leq \frac{a+\sqrt{1-a}}{a}\,\omega\left(f; \frac{1}{\sqrt{n}}\right),$$

since

$$\beta_n = \sup_I \bar{\sigma}_n(x) = \frac{1}{a}\sqrt{\frac{1-a}{n}}.$$

4°. In the case of the Baskakov operator (1.13) we take $I = [0, a]$ where $0 < a < \infty$. Since

$$\sigma_n(x) = \sqrt{\frac{x(1+x)}{n}}, \qquad \beta_n = \sqrt{\frac{a(1+a)}{n}}$$

the inequality (3.3) yields

$$|f(x) - P_n(f;x)| \leq \left(1+\sqrt{a(1+a)}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right).$$

5°. In the case of operator (1.15) we have $\sigma_n(x) = \sigma/\sqrt{n}$, for any real value of x; consequently

$$|f(x) - P_n^{(\sigma)}(f;x)| \leq (1+\sigma)\,\omega\left(f; \frac{1}{\sqrt{n}}\right),$$

where σ is a positive constant.

6°. For the operator (1.16) we take $I = (0, a)$, where $0 < a < \infty$, and we have

$$\sigma_n(x) = \frac{x}{\sqrt{n}}, \qquad \beta_n = \frac{a}{\sqrt{n}}.$$

Therefore

$$|f(x) - P_n(f;x)| \leq (1+a)\,\omega\left(f; \frac{1}{\sqrt{n}}\right).$$

7°. For our operator (1.18) we have $I = [0, 1]$ and

$$\sigma_n(x) = \sqrt{\frac{1 + \alpha n}{1 + \alpha} \cdot \frac{x(1 - x)}{n}}, \qquad \beta_n = \frac{1}{2}\sqrt{\frac{1 + \alpha n}{n + \alpha n}},$$

α being a non-negative constant.

Hence if in (3.6) we take $\gamma = 2$ and $b_n = \beta_n$, then we get the inequality

$$|f(x) - P_n^{(\alpha)}(f; x)| \leqq \frac{3}{2}\,\omega\left(f; \sqrt{\frac{1 + \alpha n}{n + \alpha n}}\right),$$

first obtained by us in [26].

5. Now let us consider the case when the function f possesses a bounded uniformly continuous derivative on the real axis.

We may state

THEOREM 3.2. *If f has a bounded uniformly continuous derivative on the real axis, then we have*

$$(3.7) \qquad |f(x) - P_n(f; x)| \leqq \beta_n(1 + \beta_n\sqrt{n})\,\omega\left(f'; \frac{1}{\sqrt{n}}\right),$$

where $\beta_n = \sup \sigma_n(x)$ for $x \in I$.

Proof. According to the mean value theorem of differential calculus we have

$$f(x) - f(y) = (x - y)f'(\xi) = (x - y)f'(x) + (x - y)\,[f'(\xi) - f'(x)],$$

where ξ is an interior point of the interval determined by x and y.

Using this equality we may write

$$f(x) - P_n(f; x) = \int_{-\infty}^{\infty}(x - y)\,[f'(\xi) - f'(x)]\,\mathrm{d}_yF_n(y; x),$$

since

$$\int_{-\infty}^{\infty} y\,\mathrm{d}_yF_n(y; x) = x.$$

Hence

$$|f(x) - P_n(f; x)| \leqq \int_{-\infty}^{\infty}|x - y|\,|f'(\xi) - f'(x)|\,\mathrm{d}_yF_n(y; x).$$

Applying the inequality (3.2) we obtain

$$|f'(\xi) - f'(x)| \leqq \omega(f'; |\xi - x|) \leqq \omega(f'; |y - x|) \leqq$$
$$\leqq (1 + \delta^{-1}|y - x|)\,\omega(f'; \delta) \qquad (\delta > 0).$$

Therefore we may write

$$|f(x) - P_n(f;x)| \leqq \left[\int_{-\infty}^{\infty} |x-y| \, d_yF_n(y;x) + \delta^{-1}\int_{-\infty}^{\infty} (x-y)^2 \, d_yF_n(y;x)\right] \omega(f';\delta).$$

It follows that

$$|f(x) - P_n(f;x)| \leqq [\sigma_n(x) + \delta^{-1}\sigma_n^2(x)] \, \omega(f';\delta),$$

and then

$$|f(x) - P_n(f;x)| \leqq \beta_n(1 + \delta^{-1}\beta_n) \, \omega(f';\delta),$$

since $\sigma_n(x) \leqq \beta_n$.

Now if we take $\delta = n^{-\frac{1}{2}}$ we obtain just the inequality (3.7) and if we let $\delta = \gamma\beta_n$ $(\gamma > 0)$ we get

$$|f(x) - P_n(f;x)| \leqq \beta_n\left(1 + \frac{1}{\gamma}\right)\omega\left(f';\gamma\beta_n\right). \tag{3.8}$$

Illustrations. Making use of the inequality (3.7) we find for the operators (1.8), (1.9). (1.11), (1.13), (1.15), (1.16) respectively the following inequalities

$$|f(x) - B_n(f;x)| \leqq \frac{3}{4}\frac{1}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right),$$

$$|f(x) - P_n(f;x)| \leqq \frac{a+\sqrt{a}}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right),$$

$$\left|f\left(\frac{1-x}{x}\right) - P_n(f;x)\right| \leqq \frac{\sqrt{1-a}}{a}\left(1 + \frac{\sqrt{1-a}}{a}\right)\frac{1}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right),$$

$$|f(x) - P_n(f;x)| \leqq \sqrt{a(1+a)}\,(1 + \sqrt{a(1+a)})\frac{1}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right),$$

$$|f(x) - P_n^{(\sigma)}(f;x)| \leqq \frac{\sigma(1+\sigma)}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right),$$

$$|f(x) - P_n(f;x)| \leqq \frac{a(1+a)}{\sqrt{n}}\,\omega\left(f';\frac{1}{\sqrt{n}}\right).$$

The first of these inequalities is due to Lorentz [14] and the second one was obtained first in [26].

In the case of our operators (1.18) it is convenient to make use of the inequality (3.8). Putting $\gamma = 2$ and inserting the corresponding value of β_n we find the following inequality

$$|f(x) - P_n^{[\alpha]}(f;x)| \leq \frac{3}{4}\sqrt{\frac{1+\alpha n}{n+\alpha n}}\,\omega\left(f';\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right),$$

which was recently established by us also in the paper [26].

Received October 19, 1968

University „Babeş Bolyai"
Faculty of Mathematics and Mechanics
Cluj, Romania

REFERENCES

1. Baskakov V. A., *An example of a sequence of linear positive operators in the space o continuous functions.* Dokl. Akad. Nauk SSSR, 1957, **113**, 249–251.
2. Bernstein S. N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités.* Commun. Soc. Math. Kharkov, 1912–1913, (2) **13**, 1–2.
3. Bohlmann G., *Formulierung und Begründung zweier Hilfssätze der mathematischen Statistik* Math. Ann., 1913, **74**, 341–409.
4. Butzer P. L. and Berens, H., *Semi-Groups of Operators and Approximation.* Berlin, Springer-Verlag, 1967.
5. Cheney E. W. and Sharma A., *Bernstein power series.* Canad. J. Math, 1964, **16**, 241–252.
6. Favard J., *Sur les multiplicateurs d'interpolation.* J. Math. Pures Appl., 1944 **23**, 219–247.
7. Feller W., *An Introduction to Probability Theory and Its Applications.* Vol. I, 2nd edition and Vol. II. New York, London, John Wiley, 1957 and resp. 1966.
8. Fisher R. A., *The negative binomial distribution.* Ann. Eugenics, 1941, **11**, 182–187.
9. Fisz M., *Probability Theory and Mathematical Statistics* (3rd edition), New York, London, John Wiley, 1963.
10. Гончаров В. Л., *Теория интерполирования и приближения функций*, Moscow, Гос. издат. техн. – теор. – лит., 1954.
11. Grüss G., *Über das Maximum des absoluten Betrages...* Math. Zeitschr. 1935, **39**, 215–226.
12. Kendall M. G. and Stuart A., *The Advanced Theory of Statistics.* Vol. **1**, *Distribution Theory* (2nd edition). London, Charles Griffin, 1962.
13. Lamperti J., *Probability: A Survey of the Mathematical Theory.* New York, Amsterdam, W. A. Benjamin, 1966.
14. Lorentz, G. G., *Bernstein Polynomials.* Toronto, Univ. of Toronto Press, 1953.
15. Lupaş A. and Müller M., *Approximationseigenschaften der Gammaoperatoren.* Math. Zeitschr. 1967, **98**, 208–226.
16. Markov A. A., *O nekotorîh predelnîh formulah iscislenia veroiatnostei*, Izvestiah Akad, Nauk, 1917, Ser. VI, **11**, 177–186.
17. Meyer-König W. and Zeller, K., *Bernsteinsche Potenzreihen.* Studia Math., 1960, **19**, 89–94.
18. Mirakyan G., *Approximation des fonctions continues au moyen de polynomes de la forme...* Dokl. Akad. Nauk SSSR, 1941, **31**, 201–205.
19. Müller M., *Die Folge der Gammaoperatoren.* Dissertation, Stuttgart, 1967.
20. Müller M., *Punktweise und gleichmässige Approximation durch Gammaoperatoren.* Math. Zeitschr., 1968, **103**, 227–238.

21. Onicescu O., *Teoria probabilităţilor şi aplicaţii.* Bucharest, Editura didactică şi pedagogică, 1963.

22. Parzen E., *Modern Probability Theory and its Applications.* New York, John Wiley, 1960.

23. Pólya G. and Szegö G., *Aufgaben und Lehrsätze aus der Analysis*, vol. I. Berlin, Springer, 1925.

24. Pólya G., *Sur quelques points de la théorie des probabilités.* Ann. Inst. H. Poincaré, 1930–1931, **1**, 117–161.

25. Popoviciu T., *Sur l'approximation des fonctions convexes d'ordre supérieur.* Mathematica, 1935, **10**, 49–54.

26. Stancu D. D., *Approximation of functions by a new class of linear polynomial operators.* Rev. Roumaine Math. Pures Appl., 1968, **13**, 1173–1194.

27. – *On the Markov-Pólya probability distribution.* Bull. Math. Soc. Sci. Math. R.S. Roumanie, 1968, **12** (**60**).

28. Szasz O., *Generalization of S. Bernstein's polynomials to the infinite interval.* J. Res. Nat. Bur. Standards, 1950, **45**, 239–245.

29. Weierstrass K., *Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reelen Veränderlichen.* Sitz. ber. Akad. Wiss. Berlin, 1885, 633–639, 789–805.