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PROBABILISTIC METHODS IN THE THEORY OF APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY LINEAR POSITIVE OPERATORS

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1. Introduction

Let $Y_n = \{(Y_{n1},\ldots,Y_{ns})\}$ be a sequence of s-dimensional random vectors and let $F_n(y_1,\ldots,y_s;x_1,\ldots,x_s)$ denote the probability distribution function of Y_n, where $(y_1,\ldots,y_s)$ is any point of the Euclidean space R_s and $(x_1,\ldots,x_s)$ is a real s-dimensional parameter, varying in a parameter space I_s, which is a subset of R_s.

We shall assume that $(x_1,\ldots,x_s)$ represents the mean value of this distribution function, i.e.,

$$x_r = \int_{R_s} y_r dF_n(y_1,\ldots,y_s;x_1,\ldots,x_s),$$

where r=1,2,...,s and the s-fold integral is extended over all $(y_1,\ldots,y_s) \in R_s$.

Consider a real-valued function $f(y_1,\ldots,y_s)$ defined and bounded on R_s, such that the mean value of the random variable $f(Y_{n1},\ldots,Y_{ns})$ exists for n=1,2,... . This mean value is given by

$$E[f(Y_{n1},\ldots,Y_{ns})] = P_n(f;x_1,\ldots,x_s)$$
$$= \int_{R_s} f(y_1,\ldots,y_s)dF_n(y_1,\ldots,y_s;x_1,\ldots,x_s). \quad (1)$$

Assuming first that the random vector Y_n is of discrete type, one may observe that its distribution function:

$$F_n(y_1,\ldots,y_s;x_1,\ldots,x_s) = \Pr[Y_{n1} \leq y_1,\ldots,Y_{ns} \leq y_s;x_1,\ldots,x_s]$$

is a step function so that $Pr[Y_{n1} = y_1,\dots,Y_{ns} = y_s; x_1,\dots,x_s]$ is zero at every point of R_s except at a finite or countably infinite number of points $(a_{r1},\dots,a_{rs})$ $(r=1,2,\dots)$ and every such point (jump point) is taken with a positive probability (jump):

$$p_n(a_{r1},\dots,a_{rs};x_1,\dots,x_s) = Pr[Y_{n1} = a_{r1},\dots,Y_{ns} = a_{rs};x_1,\dots,x_s],$$

satisfying the condition

$$\sum_r p_n(a_{r1},\dots,a_{rs};x_1,\dots,x_s) = 1.$$

The corresponding distribution function is

$$F_n(y_1,\dots,y_s;x_1,\dots,x_s) = \sum_{(r)} p_n(a_{r1},\dots,a_{rs};x_1,\dots,x_s), \tag{2}$$

where the summation is extended now over all points $(a_{r1},\dots,a_{rs})$ such that $a_{r1} \leqslant y_1,\dots,a_{rs} \leqslant y_s$.

Consequently, in this discrete case we are able to write down the following expression for the operator (1):

$$P_n(f;x_1,\dots,x_s) = \sum_r f(a_{r1},\dots,a_{rs})p_n(a_{r1},\dots,a_{rs};x_1,\dots,x_s). \tag{3}$$

If the s-dimensional random vectors Y_n are of continuous type, having the probability densities $\rho_n(y_1,\dots,y_s;x_1,\dots,x_s)$ $(n=1,2,\dots)$, then we have

$$F_n(y_1,\dots,y_s;x_1,\dots,x_s) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_s} \rho_n(u_1,\dots,u_s;x_1,\dots,x_s)du_1\dots du_s,$$

and the operator (1) can be written as follows

$$P_n(f;x_1,\dots,x_s) = \int_{R_s} f(y_1,\dots,y_s)\rho_n(y_1,\dots,y_s;x_1,\dots,x_s)dy_1\dots dy_s. \tag{4}$$

It is easy to see that the operator $P_n(f;x_1,\dots,x_s)$ defined by (1), or in particular by (3) and (4), is a positive linear operator.

2. A limit theorem

We shall henceforth assume that for $n=1,2,\dots$ the variances of the random vectors Y_n exist and that the components,

$$\sigma_{n,j}^2 = \int_{R_s} (y_j - x_j)^2 dF_n(y_1,\dots,y_s;x_1,\dots,x_s),$$

of these variances are different from zero $(j=1,2,\dots,s)$.

Now we shall state and prove the central result of this paper:

If the real-valued function $f(x_1,\ldots,x_s)$ is continuous on a compact subset D_s of Ω_s and we have uniformly on D_s

$$\lim_{n\to\infty} \sigma^2_{n,j} = 0 \quad (j=1,2,\ldots,s), \tag{5}$$

then

$$\lim_{n\to\infty} P_n(f;x_1,\ldots,x_s) = f(x_1,\ldots,x_s)$$

uniformly on D_s.

In order to prove this assertion we first take into account that since

$$P_n(1;x_1,\ldots,x_s) = \int_{R_s} dF_n(y_1,\ldots,y_s;x_1,\ldots,x_s) = 1,$$

we have

$$R_n(f;x_1,\ldots,x_s) = f(x_1,\ldots,x_s) - P_n(f;x_1,\ldots,x_s)$$
$$= \int_{R_s} [f(x_1,\ldots,x_s)-f(y_1,\ldots,y_s)]\,dF_n(y_1,\ldots,y_s;x_1,\ldots,x_s).$$

Consequently we are able to write

$$|R_n(f;x_1,\ldots,x_s)| \leq \int_{R_s} |f(x_1,\ldots,x_s)-f(y_1,\ldots,y_s)|\,dF_n(y_1,\ldots,y_s;x_1,\ldots,x_s).$$

Now it is convenient to make use of the modulus of continuity of f, defined as follows:

$$\omega(f;\delta_1,\ldots,\delta_s) = \sup|f(x''_1,\ldots,x''_s)-f(x'_1,\ldots,x'_s)|,$$

where $(x'_1,\ldots,x'_s)$ and $(x''_1,\ldots,x''_s)$ are points from D_s such that: $|x''_1 - x'_1| \leq \delta_1,\ldots,|x''_s - x'_s| \leq \delta_s$, $\delta_1,\ldots,\delta_s$ being positive numbers.

Using the following properties of the modulus of continuity:

$$|f(x''_1,\ldots,x''_s)-f(x'_1,\ldots,x'_s)| \leq \omega(f;|x''_1-x'_1|,\ldots,|x''_s-x'_s|),$$

$$\omega(f;\lambda_1\delta_1,\ldots,\lambda_s\delta_s) \leq (1 + \lambda_1 +\ldots+ \lambda_s)\omega(f;\delta_1,\ldots,\delta_s),$$

where $\lambda_1 > 0,\ldots,\lambda_s > 0$, we have

$$|f(x_1,\ldots,x_s)-f(y_1,\ldots,y_s)| \leq \omega(f;\ \frac{1}{\delta_1}|x_1-y_1|\delta_1,\ldots,\frac{1}{\delta_s}|x_s-y_s|\delta_s)$$
$$\leq (1 + \frac{1}{\delta_1}|x_1-y_1| +\ldots+ \frac{1}{\delta_s}|x_s-y_s|)\omega(f;\delta_1,\ldots,\delta_s)$$

and therefore

$$|R_n(f;x_1,\dots,x_s)| \leq \left(1 + \sum_{j=1}^{s} \frac{1}{\delta_j} P_n(|x_j-y_j|;x_1,\dots,x_s)\right)\omega(f;\delta_1,\dots,\delta_s).$$

In accordance with the Cauchy-Schwarz inequality we have

$$P_n(|x_j-y_j|;x_1,\dots,x_s) \leq \left(\int_{R_s} (x_j-y_j)^2 dF_n(y_1,\dots,y_s;x_1,\dots,x_s)\right)^{1/2} = \sigma_{n,j}.$$

We may therefore write

$$|R_n(f;x_1,\dots,x_s)| \leq \left(1 + \sum_{j=1}^{s} \frac{\sigma_{n,j}}{\delta_j}\right)\omega(f;\delta_1,\dots,\delta_s).$$

If we set $\delta_j = \alpha_j\sigma_{n,j}$ $(j=1,2,\dots,s)$, where $\alpha_1,\dots,\alpha_s$ are fixed positive numbers, we obtain finally

$$|R_n(f;x_1,\dots,x_s)| \leq \left(1 + \frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_s}\right)\omega(f;\alpha_1\sigma_{n,1},\dots,\alpha_s\sigma_{n,s}). \quad (6)$$

We note that with the aid of the constants $\alpha_1,\dots,\alpha_s$ it will be easy to obtain the desired inequalities corresponding to each concrete case.

Since by our hypothesis (5) the right-hand side of the last inequality tends to zero on D_s for $n \to \infty$, it follows that the sequence of operators $\{P_n(f;x_1,\dots,x_s)\}$ converges uniformly to $f(x_1,\dots,x_s)$ on D_s.

It should be remarked that this result represents an extension to the multidimensional case of a result due to Feller [4], and that in Stancu [13] we established an inequality of the form (6) when s=1.

3. Finite difference representation of operators

Now let us consider a method for representation by finite differences of the linear positive operators of discrete type.

It is convenient to make use of an interpolation polynomial of Newton-Biermann type for several variables (see Stancu [8], [10]):

$$N(f;t_1,\dots,t_s) = \sum_{0\leq i_1+\dots+i_s\leq n} \frac{(nt_1)^{[i_1]}\dots(nt_s)^{[i_s]}}{i_1!\dots i_s!} \Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0),$$

where

$$(nt_k)^{[i_k]} = nt_k(nt_k-1)\dots(nt_k-i_k+1),$$

while

$$\Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0) = \sum_{\nu_1=0}^{i_1}\dots\sum_{\nu_s=0}^{i_s}(-1)^{\nu_1+\dots+\nu_s}\binom{i_1}{\nu_1}\dots\binom{i_s}{\nu_s} f\left(\frac{i_1-\nu_1}{n},\dots,\frac{i_s-\nu_s}{n}\right) \quad (7)$$

[re]presents the finite partial difference of order $(i_1,\dots,i_s)$ of the [fun]ction f, with the steps $h_1=\dots=h_s=1/n$ and the starting point $(0,\dots,0)$. With the aid of the changes of variables $nt_k=y_k\,(k=1,2,\dots,s)$ we obtain

$$N(f;\frac{y_1}{n},\dots,\frac{y_s}{n}) = \sum_{0\leq i_1+\dots+i_s\leq n}\frac{y_1^{[i_1]}\dots y_s^{[i_s]}}{i_1!\dots i_s!}\Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0). \quad (8)$$

[No]te that this polynomial satisfies the interpolating properties

$$N(f;\frac{k_1}{n},\dots,\frac{k_s}{n}) = f(\frac{k_1}{n},\dots,\frac{k_s}{n}) \quad (9)$$

[fo]r $k_1=0,1,\dots,n;k_2=0,1,\dots,n-k_1;\dots;k_s=0,1,\dots,n-k_1-\dots-k_{s-1}$.

By using the formula (8) we can find for the mean value of the random [va]riable $N(f;\frac{y_1}{n},\dots,\frac{y_s}{n})$, where $(y_1,\dots,y_s)$ has the probability distribution [fu]nction $F_n(y_1,\dots,y_s;x_1,\dots,x_s)$, the following representation:

$$\int_{R_s} N(f;\frac{y_1}{n},\dots,\frac{y_s}{n})dF_n(y_1,\dots,y_s;x_1,\dots,x_s) = \sum_{0\leq i_1+\dots+i_s\leq n}\frac{m_{[i_1,\dots,i_s]}}{i_1!\dots i_s!}\Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0), \quad (10)$$

[in] terms of the factorial moments

$$m_{[i_1,\dots,i_s]} = \int_{R_s} y_1^{[i_1]}\dots y_s^{[i_s]}\,dF_n(y_1,\dots,y_s;x_1,\dots,x_s).$$

[Th]ese moments can be found by means of the corresponding factorial [mo]ment-generating function

$$g(t_1,\dots,t_s) = E(t_1^{y_1}\dots t_s^{y_s}) = \int_{R_s} t_1^{y_1}\dots t_s^{y_s}dF_n(y_1,\dots,y_s;x_1,\dots,x_s), \quad (11)$$

[ac]cording to the formula

$$m_{[i_1,\dots,i_s]} = \frac{\partial^{i_1+\dots+i_s}g(t_1,\dots,t_s)}{\partial t_1^{i_1}\dots t_s^{i_s}}\Bigg|_{\substack{t_1=1\\ \dots\\ t_s=1}} \quad (12)$$

It should be observed that if the random vector Y_n is of discrete type and assumes the values $(k_1,\dots,k_s)$ with the probabilities $p_{n;k_1,\dots,k_s}$, where $k_1=0,1,\dots,n;k_2=0,1,\dots,n-k_1;\dots;k_s=0,1,\dots,n-k_1-\dots-k_{s-1}$, then according to (9) the equality (10) reduces to

$$\sum_{0\le k_1+\dots+k_s\le n} p_{n;k_1,\dots,k_s}\, f\left(\frac{k_1}{n},\dots,\frac{k_s}{n}\right)$$

$$= \sum_{0\le k_1+\dots+k_s\le n} \frac{m_{[i_1,\dots,i_s]}}{i_1!\dots i_s!}\, \Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0), \quad (13)$$

where

$$m_{[i_1,\dots,i_s]} = \sum_{0\le k_1+\dots+k_s\le n} k_1^{[i_1]}\dots k_s^{[i_s]}\, p_{n;k_1,\dots,k_s}.$$

4. Operators for uniform approximation

Let us show next a special method for obtaining concrete positive linear operators suitable for uniform approximation of continuous functions.

Consider a sequence of s-dimensional random vectors $\{X_k=(X_{k1},\dots,X_{ks})\}$ and let us assume that the components $Y_{n\nu}$ of the random vector Y_n represent the arithmetic means of the first n components $X_{k\nu}$ $(k=1,2,\dots,n)$ for $\nu=1,2,\dots,s$, that is

$$Y_{n\nu} = \frac{1}{n}[X_{1\nu}+\dots+X_{n\nu}] \quad (\nu=1,2,\dots,s).$$

If we make the further assumption that for any natural number n the random vectors $X_1,\dots,X_n$ are independent and identically distributed , then the common characteristic function $\phi(t_1,\dots,t_s)$ of these random vectors, which is given by

$$\phi(t_1,\dots,t_s) = \int_{R_s} \exp\left(i\sum_{\nu=1}^{s} t_\nu x_\nu\right) dF(x_1,\dots,x_s),$$

$F(x_1,\dots,x_s)$ being the common distribution function of these random vectors, can be used for expressing the characteristic function of the random vector Y_n:

$$\psi_n(t_1,\dots,t_s) = \left[\phi\left(\frac{t_1}{n},\dots,\frac{t_s}{n}\right)\right]^n, \quad (14)$$

since

$$\sum_{\nu=1}^{s} t_\nu Y_{n\nu} = \frac{1}{n} \sum_{k=1}^{n} [t_1 X_{k1} + \ldots + t_s X_{ks}].$$

5. Examples

For illustrative purposes we shall consider some examples, assuming that the random vectors $X_m (m=1,2,\ldots)$ are independent and identically distributed.

(i) Let us suppose first that the random vector $X_m = (X_{m1},\ldots,X_{ms})$ has at most one component different from zero and

$$\Pr[X_{m\nu}=1] = x_\nu, \quad \Pr[X_{m\nu}=0] = 1-x_\nu,$$

$$\Pr[X_{m1}=0,\ldots,X_{ms}=0] = y = 1-x_1-\ldots-x_s ,$$

where $0 \leqslant x_\nu \leqslant 1 \quad (\nu=1,2,\ldots,s)$.

It follows that the characteristic function of X_m is given by

$$\phi(t_1,\ldots,t_s) = x_1 \exp(it_1)+\ldots+x_s \exp(it_s) + y.$$

Hence the characteristic function of $Y_n = (Y_{n1},\ldots,Y_{ns})$ can be expressed as follows:

$$\psi_n(t_1,\ldots,t_s) = \{x_1 \exp(\frac{it_1}{n})+\ldots+x_s \exp(\frac{it_s}{n})+y\}^n .$$

Clearly, this corresponds (see, e.g., Wilks [16]) to the random vector Y_n, whose components $Y_{n\nu}$ are $Z_{n\nu}/n$, where $Z_{n\nu} = X_{1\nu}+\ldots+X_{n\nu}$ $(\nu=1,2,\ldots,s)$, the random vector $Z_n = (Z_{n1},\ldots,Z_{ns})$ having the multinomial distribution:

$$\Pr[Z_{n1}=k_1,\ldots,Z_{ns}=k_s] = p_n^{k_1,\ldots,k_s}(x_1,\ldots,x_s)$$

$$= \frac{n!}{k_1!\ldots k_s!(n-k_1-\ldots-k_s)!} x_1^{k_1}\ldots x_s^{k_s}(1-x_1-\ldots-x_s)^{n-k_1-\ldots-k_s}.$$

Now referring to (3) we obtain the following operator

$$B_n(f;x_1,\ldots,x_s) = \sum_{0\leqslant k_1+\ldots+k_s\leqslant n} p_n^{k_1,\ldots,k_s}(x_1,\ldots,x_s) f(\frac{k_1}{n},\ldots,\frac{k_s}{n}). \quad (15')$$

It is easily seen that this operator may also be written in the form

$$B_n(f;x_1,\ldots,x_s)$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_s=0}^{n-k_1-\ldots-k_{s-1}} q_n^{k_1,\ldots,k_s}(x_1,\ldots,x_s) f(\frac{k_1}{n},\ldots,\frac{k_s}{n}), \quad (15)$$

where

$$q_n^{k_1,\dots,k_s}(x_1,\dots,x_s) \equiv p_n^{k_1,\dots,k_s}(x_1,\dots,x_s)$$
$$= \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\dots-k_{s-1}}{k_s} x_1^{k_1}\cdots x_s^{k_s}(1-x_1-\dots-x_s)^{n-k_1-\dots-k_s}.$$

It is obvious that this linear operator is positive on the s-dimensional simplex

$$\Delta_s = \{(x_1,\dots,x_s);x_1\geq 0,\dots,x_s\geq 0,x_1+\dots+x_s\leq 1\}. \tag{16}$$

Since in this case the factorial moment-generating function is

$$g(t_1,\dots,t_s) = (x_1t_1+\dots+x_st_s+y)^n,$$

we have

$$m_{[i_1,\dots,i_s]} = n^{[i_1+\dots+i_s]} x_1^{i_1}\cdots x_s^{i_s}.$$

Consequently, with the aid of formula (13) we can give the following representation of the operators of Bernstein type (15), in terms of finite differences:

$$B_n(f;x_1,\dots,x_s)$$
$$= \sum_{0\leq i_1+\dots+i_s\leq n} \frac{n^{[i_1+\dots+i_s]}}{i_1!\cdots i_s!} x_1^{i_1}\cdots x_s^{i_s} \Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0),$$

which enables us to find at once

$$\sigma^2_{n,j} = B_n((t_j-x_j)^2;x_1,\dots,x_s) = \frac{x_j(1-x_j)}{n} \leq \frac{1}{4n} \quad (j=1,2,\dots,s).$$

Therefore, in this case the inequality (6) becomes

$$|f(x_1,\dots,x_s)-B_n(f;x_1,\dots,x_s)| \leq (1+\frac{1}{\alpha_1}+\dots+\frac{1}{\alpha_s})\omega(f;\frac{\alpha_1}{2\sqrt{n}},\dots,\frac{\alpha_s}{2\sqrt{n}}).$$

Now if we choose $\alpha_1=\dots=\alpha_s=2$ we obtain finally the inequality

$$|f(x_1,\dots,x_s)-B_m(f;x_1,\dots,x_s)| \leq (1+\frac{s}{2})\omega(f;\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}),$$

valid for $(x_1,\dots,x_s) \in \Delta_s$, f being continuous on the simplex Δ_s. This inequality was given by us in 1960 [9]. In 1962 Schurer [6], using an important result due to Sikkema [7], obtained a better constant which can be used in place of $1+\frac{s}{2}$ in this inequality.

(ii) Now suppose that the random vector $X_m=(X_{m1},\dots,X_{ms})$ has an s-variate Poisson distribution, i.e.,

$$Pr(X_{m1}=k_1,\dots,X_{ms}=k_s) = p(k_1,\dots,k_s;x_1,\dots,x_s)$$

$$= e^{-(x_1+\dots+x_s)} \frac{x_1^{k_1}\dots x_s^{k_s}}{k_1!\dots k_s!} \qquad (k_1,\dots,k_s = 0,1,2,\dots).$$

The characteristic functions of X_m and $Y_n=(Y_{n1},\dots,Y_{ns})$ are respectively

$$\phi(t_1,\dots,t_s) = \exp(x_1 \exp(it_1)+\dots+x_s \exp(it_s)-x_1-\dots-x_s),$$

$$\psi_n(t_1,\dots,t_s) = \exp(nx_1 \exp(it_1/n)+\dots+nx_s \exp(it_s/n)-nx_1-\dots-nx_s).$$

It is easily seen that the latter represents at the same time the characteristic function of $Y_n = (\frac{X_1}{n},\dots,\frac{X_s}{n})$, where $(X_1,\dots,X_s)$ has an s-variate Poisson distribution with the parameters $nx_1,\dots,nx_s$.

In the case of this distribution, formula (3) leads us to the operator

$$P_n(f;x_1,\dots,x_s) = \sum_{k_1,\dots,k_s=0}^{\infty} e^{-n(x_1+\dots+x_s)} \frac{(nx_1)^{k_1}\dots(nx_s)^{k_s}}{k_1!\dots k_s!} f(\frac{k_1}{n},\dots,\frac{k_s}{n}), \tag{18}$$

which represents an extension to s variables of an operator studied early by Favard [3] and Szasz [15].

Assuming that x_j depends on n in such a way that for $n\to\infty$ we have $nx_j \to z_j > 0$ $(j=1,2,\dots,s)$, we can obtain this operator starting from the operator (15). In this limit case formula (17) permits us to find the following representation for the operator (18):

$$P_n(f;z_1,\dots,z_s) = \sum_{i_1,\dots,i_s=0} \frac{(nx_1)^{i_1}\dots(nz_s)^{i_s}}{i_1!\dots i_s!} \Delta^{i_1,\dots,i_s}_{\frac{1}{n},\dots,\frac{1}{n}} f(0,\dots,0).$$

If we presuppose that f is continuous on

$$D_s:\{(x_1,\dots,x_s):0 \le x_j \le a_j, a_j > 0,\ j=1,2,\dots s\},$$

then by virtue of (6) we obtain the inequality

$$|f(x_1,\dots,x_s)-P_n(f;x_1,\dots,x_s)| \le (1+\sqrt{a_1}+\dots+\sqrt{a_s})\omega(f;\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}),$$

valid for $(x_1,\dots,x_s) \in D_s$, since in this case $\sigma^2_{n,j} = \frac{x_j}{n} \le \frac{a_j}{n}$ and we can select $\alpha_j = 1/\sqrt{a_j}$ $(j=1,2,\dots,s)$.

(iii) We now assume that the random vector X_m has a geometric distribution:

$$\Pr(X_{m1}=k_1,\ldots,X_{ms}=k_s) = \prod_{j=1}^{s} x_j(1-x_j)^{k_j} \quad (k_1,\ldots,k_s=0,1,2,\ldots),$$

where $0 \leq x_j \leq 1$ $(j=1,2,\ldots,s)$.

Since the characteristic function of X_m is

$$\phi(t_1,\ldots,t_s) = \prod_{j=1}^{s} \frac{x_j}{1-(1-x_j)\exp(it_s)},$$

we obtain for the characteristic function of the vector Y_n the following expression

$$\psi_n(t_1,\ldots,t_s) = \prod_{j=1}^{s} \left(\frac{x_j}{1-(1-x_j)\exp(it_s/n)}\right)^n.$$

One observes that this represents the characteristic function of $Y_n = (\frac{X_1}{n},\ldots,\frac{X_s}{n})$, where $(X_1,\ldots,X_s)$ has an s-variate Pascal distribution:

$$\Pr(X_1=k_1,\ldots,X_s=k_s) = \prod_{j=1}^{s} \binom{n+k_j-1}{k_j} x_j^n(1-x_j)^{k_j},$$

where $k_1,\ldots,k_s=0,1,2,\ldots$.

In accordance with (3) we therefore obtain the operator

$$P_n(f;x_1,\ldots,x_s) = \sum_{k_1,\ldots,k_s=0}^{\infty} \left\{\prod_{j=1}^{s} \binom{n+k_j-1}{k_j} x_j^n(1-x_j)^{k_j}\right\} f(\frac{k_1}{n},\ldots,\frac{k_s}{n}). \quad (19)$$

It should be observed that if we replace x_j by $1/(1+x_j)$ $(j=1,2,\ldots,s)$ then we arrive at the operator

$$Q_n(f;x_1,\ldots,x_s) = \sum_{k_1,\ldots,k_s=0}^{\infty} \left\{\prod_{j=1}^{s} \binom{n+k_j-1}{k_j} \frac{x_j^{k_j}}{(1+x_j)^{n+k_j}}\right\} f(\frac{k_1}{n},\ldots,\frac{k_s}{n}), \quad (20)$$

which in the case s=1 has been considered first by Baskakov [1].

Since in this case of operator (19) we have

$$\overline{\sigma}_{n,j} = \left(P_n\left\{\left(\frac{1-x_j}{x_j} - y_j\right)^2;x_1,\ldots,x_s\right\}\right)^{1/2} = \frac{1}{x_j}\sqrt{(1-x_j)/n}$$

assuming that f is continuous on

$$D_s':\{(x_1,\ldots,x_s):a_j \leq x_j \leq 1,\ 0 < a_j < 1,\ j=1,2,\ldots,s\},$$

we have on D_s'

$$\left|f\left(\frac{1-x_1}{x_1},\dots,\frac{1-x_s}{x_s}\right) - P_n(f;x_1,\dots,x_s)\right| \le \left(1+\sum_{j=1}^{s}\frac{\sqrt{1-a_j}}{a_j}\right)\omega(f;\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}).$$

On the other hand, for the operator (20) we have

$$\sigma_{n,j} = \left\{Q_n\left[(x_j-y_j)^2;x_1,\dots,x_s\right]\right\}^{1/2} = \sqrt{x_j(1+x_j)/n} \quad (j=1,2,\dots,s).$$

Consequently, if we presuppose that f is continuous on

$$D''_s : \{(x_1,\dots,x_s) : 0 \le x_j \le a_j,\ a_j > 0,\ j=1,2,\dots,s\},$$

we obtain at once the inequality

$$|f(x_1,\dots,x_s) - Q_n(f;x_1,\dots,x_s)| \le \left(1+\sum_{j=1}^{s}\sqrt{a_j(1+a_j)}\right)\omega(f;\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}).$$

(iv) We shall now consider a case when the random vectors $X_1,X_2,\dots$ are not independent and identically distributed.

Let us assume that we have an urn which contains N balls : a_o of color 0, a_1 of color $1,\dots,a_s$ of color s. A ball is drawn at random and if it is of color k it is replaced with c additional balls of color $k(k=0,1,\dots,s)$. This procedure is repeated n times. Let X_{ij} be one or zero according as the i-th trial results in a ball of color j or not. The probability that the total number of balls of color $j:X_{1j}+X_{2j}+\dots+X_{nj}$ be equal to k_j, where $0 \le k_j \le n$ $(j=0,1,\dots,s)$ and $k_0+k_1+\dots+k_s=n$, is given by

$$\frac{C_n^{k_1,\dots,k_s}}{A(n)}\prod_{i=0}^{s}\prod_{\nu_i=0}^{k_i-1}(a_i+\nu_i c),$$

where

$$C_n^{k_1,\dots,k_s} = \frac{n!}{k_0!k_1!\dots k_s!} = \frac{n!}{k_1!\dots k_s!(n-k_1-\dots-k_s)!}$$

and $A(n)=N(N+c)\dots(N+\overline{n-1}c)$.

This is the probability function of the s-variate Markov-Polya distribution.

Now let: $a_j/N = x_j$ $(j=1,2,\dots,s)$, $c/N=\alpha$. Since $a_o/N=1-x_1-\dots-x_s$, it follows that the probability of

$$Y_{nj} = \frac{1}{n}\sum_{i=1}^{n}X_{ij} = \frac{k_j}{n} \quad (j=1,2,\dots,s)$$

can be written down at once as

$$w_n^{k_1,\dots,k_s}(x_1,\dots,x_s;\alpha)$$

$$= \frac{C_n^{k_1,\dots,k_s}}{B(n)} \left\{ \prod_{j=1}^{s} \prod_{\nu_j=0}^{k_j-1} (x_j+\nu_j\alpha) \right\} \prod_{\mu=0}^{n-k_1-\dots-k_s-1} (1-x_1-\dots-x_s+\mu\alpha),$$

where $B(n)=(1+\alpha)(1+2\alpha)\dots(1+\overline{n-1}\alpha)$.

It is now easy to see that in this case we obtain the operator*

$$P_n^{[\alpha]}(f;x_1,\dots,x_s) = \sum_{0\leqslant k_1+\dots+k_s\leqslant n} w_n^{k_1,\dots,k_s}(x_1,\dots,x_s;\alpha) f(\tfrac{k_1}{n},\dots,\tfrac{k_s}{n}), \quad (21)$$

which in the case s=1 was introduced and studied in detail in our papers [11], [12], [13].

It is obvious that (21) represents a polynomial of degree m with respect to $x_1,\dots,x_s$ and that for $\alpha \geqslant 0$ it is a <u>positive</u> linear operator on the simplex Δ_s defined in (16). In the special case $\alpha=0$ it reduces to the n-dimensional Bernstein polynomial† (15').

It is easy to verify that if $x_1 > 0,\dots,x_s > 0$, $x_1+\dots+x_s < 1$ and $\alpha > 0$, then we can write

$$\frac{1}{C_n^{k_1,\dots,k_s}} w_n^{k_1,\dots,k_s}(x_1,\dots,x_s;\alpha)$$

$$= \frac{B(\frac{x_1}{\alpha}+k_1,\dots,\frac{x_s}{\alpha}+k_s, \frac{1-x_1-\dots-x_s}{\alpha}+n-k_1-\dots-k_s)}{B(\frac{x_1}{\alpha},\dots,\frac{x_s}{\alpha}, \frac{1-x_1-\dots-x_s}{\alpha})},$$

by using the s-fold Dirichlet integral

$$B(p_1,\dots,p_{s+1}) = \int_{\Delta_s} u_1^{p_1-1}\dots u_s^{p_s-1}(1-u_1-\dots-u_s)^{p_{s+1}-1} du_1\dots du_s. \quad (22)$$

By taking this into consideration, we readily arrive at the following representation of the corresponding factorial moment-generating function (11):

* In [14] we investigated another operator of this type.

† See Dinghas [2], Lorentz [5] and Stancu [9].

$$B\left(\frac{x_1}{\alpha},\ldots,\frac{x_s}{\alpha},\frac{1-x_1-\ldots-x_s}{\alpha}\right)g(t_1,\ldots,t_s) =$$

$$u_1^{\frac{x_1}{\alpha}-1}\ldots u_s^{\frac{x_s}{\alpha}-1}(1-u_1-\ldots-u_s)^{\frac{1-x_1-\ldots-x_s}{\alpha}-1}(1-u_1-\ldots-u_s+t_1u_1+\ldots+t_su_s)^n du_1\ldots du_s.$$

Consequently, if we take into account formula (12) and that the Dirichlet integral (22) has the value (see, e.g., Wilks [16]):

$$B(p_1,\ldots,p_{s+1}) = \frac{\Gamma(p_1)\ldots\Gamma(p_{s+1})}{\Gamma(p_1+\ldots+p_{s+1})},$$

we obtain at once the following expressions for the factorial moments of the s-variate Markov-Polya distribution:

$$m_{[i_1,\ldots,i_s]} = n^{[i_1+\ldots+i_s]}\frac{B\left(\frac{x_1}{\alpha}+i_1,\ldots,\frac{x_s}{\alpha}+i_s,\frac{1-x_1-\ldots-x_s}{\alpha}\right)}{B\left(\frac{x_1}{\alpha},\ldots,\frac{x_s}{\alpha},\frac{1-x_1-\ldots-x_s}{\alpha}\right)} =$$

$$= \frac{n^{[i_1+\ldots+i_s]}}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{i_1+\ldots+i_s-1}\alpha)}\prod_{\nu=1}^{s} x_\nu(x_\nu+\alpha)\ldots(x_\nu+\overline{i_\nu-1}\alpha).$$

Now referring to (13) we can give the following representation for our operator (21):

$$P_n^{[\alpha]}(f;x_1,\ldots,x_s) = \sum_{0\leq i_1+\ldots+i_s\leq n} C_n^{i_1,\ldots,i_s}\frac{\prod_{\nu=1}^{s} x_\nu(x_\nu+\alpha)\ldots(x_\nu+\overline{i_\nu-1}\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{i_1+\ldots+i_s-1}\alpha)}\Delta^{i_1,\ldots,i_s}_{\frac{1}{n},\ldots,\frac{1}{n}}f(0,\ldots,0),$$

by means of the finite partial differences (7).

The application of this formula yields

$$P_m^{[\alpha]}(1;x_1,\ldots,x_s) = 1,\quad P_m^{[\alpha]}(y_j;x_1,\ldots,x_s) = x_j,$$

$$P_m^{[\alpha]}(y_j^2;x_1,\ldots,x_s) = \frac{1}{1+\alpha}\left\{\frac{x_j(1-x_j)}{n} + x_j(x_j+\alpha)\right\},$$

so that

$$\sigma_{n,j}^2 = \frac{1+\alpha n}{1+\alpha}\cdot\frac{x_j(1-x_j)}{n}\quad (j=1,2,\ldots,s).$$

The inequality (6), where we take $\alpha_1=\ldots=\alpha_s=2$, enables us to write down

$$|f(x_1,\dots,x_s)-P_n^{[\alpha]}(f;x_1,\dots,x_s)| \leq (1+\frac{s}{2})\omega\left(f;\sqrt{\frac{1+\alpha n}{n+\alpha n}},\dots,\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right). \tag{23}$$

Consequently, if $f(x_1,\dots,x_s)$ is continuous on Δ_s and if $0 \leq \alpha = \alpha(n) \to 0$ as $n \to \infty$, then the sequence of operators $\{P_n^{[\alpha]}(f;x_1,\dots,x_s)\}$ converges to $f(x_1,\dots,x_s)$ uniformly on Δ_s.

It should be noticed that when α=0 the inequality (23) reduces to one established by us [9] for the s-dimensional Bernstein operator (15).

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