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APPROXIMATION OF FUNCTIONS BY MEANS OF SOME NEW CLASSES OF POSITIVE LINEAR OPERATORS

by D.D. Stancu in Cluj

The purpose of this paper is to introduce some new classes of positive linear operators, depending on some real parameters, and to examine their main approximation properties to real-valued functions. These operators generalize the well-known operators of Bernstein, Baskakov, Favard-Szasz, etc.

§ 1. APPROXIMATION OF FUNCTIONS BY MEANS OF A CLASS OF LINEAR POLYNOMIAL POSITIVE OPERATORS

1. Let us consider a sequence of polynomials depending on a real parameter α: $(\varphi_m^{\langle\alpha\rangle})_{m=1}^{\infty}$, where $\varphi_m^{\langle\alpha\rangle}$ is of degree m.

According to the Gregory-Newton interpolation formula, using the $m+1$ equally spaced points: $x, x+\alpha, \ldots, x+m\alpha$, where x is any fixed point of an interval $I=[o,a]$ $(a>o)$, one can give the representation

$$\varphi_m^{\langle\alpha\rangle}(y)=\sum_{k=o}^{m}(y-x)^{(k,\alpha)}\frac{\Delta_\alpha^k\varphi_m^{\langle\alpha\rangle}(x)}{k!\,\alpha^k},$$

where Δ_α is the symbol for forward-differences with the step α, and we have used the notation $u^{(\nu,h)}$ for the factorial power of order ν and increment h of u, namely

$$u^{(\nu,h)} = u(u-h)\dots(u-\overline{\nu-1}h), \quad u^{(o,h)} = 1 \quad (u \neq o).$$

If we introduce the Nörlund difference quotient D_α^k, defined by

$$D_\alpha^k g(x) = D_\alpha(D_\alpha^{k-1} g(x)), \quad D_\alpha g(x) = \frac{g(x+\alpha)-g(x)}{\alpha}, \quad D_\alpha^o g(x) = g(x),$$

and take $y = o$ we obtain

$$\varphi_m^{\langle\alpha\rangle}(o) = \sum_{k=o}^{m} \frac{(-x)^{(k,\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x),$$

since

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = \alpha^{-k} \Delta_\alpha^k \varphi_m^{\langle\alpha\rangle}(x).$$

Assuming that $\varphi_m^{\langle\alpha\rangle}(o) \neq o$ and observing that $(-x)^{(k,\alpha)} = (-1)^k x^{(k,-\alpha)}$, we can write the identity

$$(1.1) \qquad (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{k=o}^{m} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = 1.$$

We now associate to each function f defined on an interval J, such that $I \subseteq J$, a linear operator $L_m^{\langle\alpha\rangle}$, depending on the parameter α, given by

$$(1.2) \qquad (L_m^{\langle\alpha\rangle} f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{k=o}^{m} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \cdot f(x_{m,k}),$$

where $x_{m,k} \in J$ for each $k = o(1)m$ and $m = 1, 2, \dots$.

Evidently, we have: $(L_m^{\langle\alpha\rangle} f)(o) = f(x_{m,o})$.

Presupposing that $\alpha \geq o$ and that for each $m = 1, 2, \dots$ the polynomials $\varphi_m^{\langle\alpha\rangle}$ satisfy the conditions:

$$\varphi_m^{\langle\alpha\rangle}(o) > o, \qquad (-1)^k D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \geq o$$

for $k = o(1)m$ and $x \in I$, one sees that $(L_m^{\langle\alpha\rangle})$ represents a sequence of *positive linear polynomial operators*.

It is readily seen that if $\alpha > o$, then we can write

$$(1.2') \qquad (L_m^{\langle\alpha\rangle} f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{k=o}^{m} \binom{-x/\alpha}{k} \Delta_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \cdot f(x_{m,k}).$$

The operators $L_m^{\langle\alpha\rangle}$ can be suitable for uniform approximation of functions $f \in C[o,a]$. They generalize some important linear positive operators used in approximation theory of functions.

Examples. If we take

$$(1.3) \qquad I = J = [o,1], \quad x_{m,k} = \frac{k}{m}, \quad \varphi_m^{\langle\alpha\rangle}(x) = (1-x)^{(m,-\alpha)}$$

by applying the following known formula

$$(1.4) \qquad \Delta_\alpha^k u^{(r,\alpha)} = r^{(k,1)} \alpha^k u^{(r-k,\alpha)}$$

we have

$$\Delta_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^m \Delta_\alpha^k (x-1)^{(m,\alpha)} = (-1)^k \alpha^k m^{(k,1)} (1-x)^{(m-k,-\alpha)},$$

so that we obtain the operator $L_m^{\langle\alpha\rangle}$ defined by

$$(1.5) \qquad (L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=o}^{m} \binom{m}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}}{1^{(m,-\alpha)}} f\left(\frac{k}{m}\right),$$

introduced and investigated in detail in our previous papers [10], [11], [12], [14], [16].

If we make in (1.5): $\alpha \to o$, we get the mth Bernstein operator:

$$(B_m f)(x) = \sum_{k=o}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right).$$

On the other hand, if we denote $\varphi_m^{\langle o\rangle}$ by φ_m, we can write

$$\lim_{\alpha \to o} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = \lim_{\alpha \to o} \frac{\Delta_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)}{\alpha^k} = \varphi_m^{(k)}(x),$$

so that formula (1.2) leads us to the definition of the operator L_m given by

$$(1.6)\qquad (L_m f)(x) = (1/\varphi_m(o)) \sum_{k=o}^{m} (-1)^k \frac{x^k}{k!} \varphi_m^{(k)}(x) f(x_{m,k}).$$

By taking

$$(1.7)\qquad I = J = [o,1], \qquad \varphi_m(x) = (1-x)^m, \qquad x_{m,k} = \frac{k+\beta}{m+\gamma},$$

where $o \le \beta \le \gamma$, one obtains the generalization of the Bernstein operator investigated in our paper [13].

If $\varphi_m(o) = 1$ and $x_{m,k} = k/m$ then at (1.6) we have the polynomial operator of BASKAKOV [1].

2. In the following we shall establish some important properties of the operators $L_m^{\langle\alpha\rangle}$.

First we prove

LEMMA 1. *The difference quotient of order r $(o \le r \le m)$, with the step $-\alpha$ and the starting point $x \in [o,1]$, of $(L_m^{\langle\alpha\rangle} f)(x)$, corresponding to $x_{m,k} = k/m$, can be represented in the following form*

$$(1.8)\qquad D_{-\alpha}^r (L_m^{\langle\alpha\rangle} f)(x) = ((-1)^r/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{k=o}^{m-r} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha}^r \varphi_m^{\langle\alpha\rangle}(x)) \Delta_{\frac{1}{m}}^r f(o).$$

Proof. We shall proceed by induction on r. If $r = 1$ we have

$$\varphi_m^{\langle\alpha\rangle}(o) D_{-\alpha}(L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=o}^{m} (-1)^k \frac{1}{k!} D_{-\alpha}[x^{(k,-\alpha)} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)]\, f(\frac{k}{m}).$$

By using the well known formula

$$(1.9)\qquad D_{-\alpha}^p u(x) v(x) = \sum_{j=o}^{p} \binom{p}{j} D_{-\alpha}^j u(x) \cdot D_{-\alpha}^{p-j} v(x - j\alpha),$$

we can write

$$D_{-\alpha}[x^{(k,-\alpha)} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)] = x^{(k,-\alpha)} D_{-\alpha}(D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)) + $$
$$+ k x^{(k-1,-\alpha)} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x-\alpha),$$

since

$$D_{-\alpha} x^{(k,-\alpha)} = k x^{(k-1,-\alpha)}.$$

Hence we obtain

$$\varphi_m^{\langle\alpha\rangle}(o) D_{-\alpha}(L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=o}^{m} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x)) f(\frac{k}{m}) +$$
$$+ \sum_{k=1}^{m} (-1)^k \frac{x^{(k-1,-\alpha)}}{(k-1)!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x-\alpha) f(\frac{k}{m}).$$

If we take into account that

$$D_\alpha^m (D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x)) = o$$

and that

$$(1.10)\qquad D_\alpha \varphi_m^{\langle\alpha\rangle}(x-\alpha) = D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x),$$

by setting $k-1 = j$ in the second sum and then denoting again the summation variable by k, we obtain

$$\varphi_m^{\langle\alpha\rangle}(o) D_{-\alpha}(L_m^{\langle\alpha\rangle} f)(x) = \sum_{k=o}^{m-1} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x)) f(\frac{k}{m}) -$$
$$- \sum_{k=o}^{m-1} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x)) f(\frac{k+1}{m}) =$$
$$= - \sum_{k=o}^{m-1} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha} \varphi_m^{\langle\alpha\rangle}(x))\, \Delta_{\frac{1}{m}} f(\frac{k}{m}),$$

so that formula (1.8) is true for $r = 1$.

Now let us assume that $r > 1$ and that the equality (1.8) is valid for $r - 1$ namely

$$(1.11)\quad \varphi_m^{\langle\alpha\rangle}(0)D_{-\alpha}^{r-1}(L_m^{\langle\alpha\rangle}f)(x) =$$

$$= (-1)^{r-1}\sum_{k=0}^{m-r+1}(-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k(D_{-\alpha}^{r-1}\varphi_m^{\langle\alpha\rangle}(x))\Delta_{1/m}^{r-1}f(\frac{k}{m}).$$

Since according to formulas (1.9) and (1.10) we have

$$D_{-\alpha}\{x^{(k,-\alpha)}D_\alpha^k(D_{-\alpha}^{r-1}\varphi_m^{\langle\alpha\rangle}(x))\} =$$

$$= x^{(k,-\alpha)}D_\alpha^k(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x)) + kx^{(k-1,-\alpha)}D_\alpha^k(D_{-\alpha}^{r-1}\varphi_m^{\langle\alpha\rangle}(x-\alpha))$$

$$= x^{(k,-\alpha)}D_\alpha^k(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x)) + kx^{(k-1,-\alpha)}D_\alpha^{k-1}(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x)),$$

by applying the operator D_α to both members of the equality (1.11) we obtain

$$\varphi_m^{\langle\alpha\rangle}(0)D_{-\alpha}^{r}(L_m^{\langle\alpha\rangle}f)(x) =$$

$$= (-1)^{r-1}\sum_{k=0}^{m-r}(-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x))\Delta_{1/m}^{r-1}f(\frac{k}{m}) +$$

$$+ (-1)^{r-1}\sum_{k=0}^{m-r+1}(-1)^k \frac{x^{(k-1,-\alpha)}}{(k-1)!} D_\alpha^{k-1}(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x))\Delta_{1/m}^{r-1}f(\frac{k}{m}).$$

Since by setting $k-1 = j$ in the last sum we obtain

$$(-1)^{r-1}\sum_{j=0}^{m-r}(-1)^{j+1} \frac{x^{(j,-\alpha)}}{j!} D_\alpha^j(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x))\Delta_{1/m}^{r-1}f(\frac{j+1}{m}),$$

it follows that we can write finally

$$\varphi_m^{\langle\alpha\rangle}(0)\cdot D_{-\alpha}^{r}(L_m^{\langle\alpha\rangle}f)(x) =$$

$$= (-1)^{r-1}\sum_{k=0}^{m-r}(-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k(D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x))\{\Delta_{1/m}^{r-1}f(\frac{k}{m})-\Delta_{1/m}^{r-1}f(\frac{k+1}{m})\} =$$

$$= (-1)^{r}\sum_{k=0}^{m-r}(-1)^k \frac{x^{(k,-\alpha)}}{k!} (D_{-\alpha}^{r}\varphi_m^{\langle\alpha\rangle}(x))\cdot\Delta_{1/m}^{r}f(\frac{k}{m}).$$

Hence formula (1.8) is valid for any natural number r such that $1 \le r \le m$.

Employing Lemma 1 we can prove an important theorem concerning the representation of the polynomial $L_m^{\langle\alpha\rangle}f$ by means of the forward-differences of the function f on the nodes used.

THEOREM 1.1. *For any $x \in [0,1]$ the polynomial $L_m^{\langle\alpha\rangle}f$, using the nodes $x_{m,k} = k/m$, can be expressed in the following form*

$$(1.12)\quad (L_m^{\langle\alpha\rangle}f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(0))\sum_{j=0}^{m}(-1)^j \frac{x^{(j,-\alpha)}}{j!}(D_{-\alpha}^{j}\varphi_m^{\langle\alpha\rangle}(0))\cdot\Delta_{1/m}^{j}f(0).$$

Proof. According to Gregory-Newton's interpolation formula, corresponding to the polynomial $L_m^{\langle\alpha\rangle}f$ and the nodes $-k\alpha$, $k = 0(1)m$, we can write

$$(L_m^{\langle\alpha\rangle}f)(x) = \sum_{j=0}^{m}\frac{x^{(j,-\alpha)}}{j!}D_{-\alpha}^{j}(L_m^{\langle\alpha\rangle}f)(0).$$

But if we set in (1.8): $r = j$ and $x = 0$ we obtain

$$D_{-\alpha}^{j}(L_m^{\langle\alpha\rangle}f)(0) = ((-1)^j/\varphi_m^{\langle\alpha\rangle}(0))(D_{-\alpha}^{j}\varphi_m^{\langle\alpha\rangle}(0))\cdot\Delta_{1/m}^{j}f(0)$$

and the insertion of it in the preceding equality leads us just to formula (1.12). If we make in (1.12) $\alpha \to 0$ and denote $\varphi_m^{\langle 0\rangle}$ by φ_m, then it follows that the polynomial given at (1.6) can be represented in the form

$$(1.13)\quad (L_m f)(x) = (1/\varphi_m(0))\sum_{j=0}^{m}(-1)^j \frac{x^j}{j!}\varphi_m^{(j)}(0)\cdot\Delta_{1/m}^{j}f(0).$$

Remark 1.1. We wish to note that, in the special case (1.3), in the paper [10] we have proved formula (1.12) by making use of the representation of the polynomial $L_m^{\langle\alpha\rangle} f$ by means of the beta function, considering that $\alpha > o$. In the next paper [12] we gave a proof of (1.12), without any restriction on α, by using the factorial moments of the Markov - Pólya probability distribution. Formula (1.13) for the choices (1.7) can be seen in our paper [13]; by replacing in it $\beta = \gamma = o$ one obtains a known representation of the Bernstein polynomials (see, e.g., LORENTZ [3]).

As a straightforward consequence of Theorem 1.1. we have

COROLLARY 1.1. *If* f *is a polynomial of degree* p, *where* $o \leq p \leq m$, *then* $L_m^{\langle\alpha\rangle} f$ *is a polynomial of degree not exceeding* p.

Since by virtue of a known relationship between a divided difference on equally spaced points and the corresponding forward-difference we can write

$$\Delta^j_{1/m} f(o) = j!\, m^{-j} [o, \tfrac{1}{m}, \tfrac{2}{m}, \dots, \tfrac{j}{m}; f],$$

it follows that we can formulate

COROLLARY 1.2. *For any* $x \in [o,1]$ *the polynomial* $L_m^{\langle\alpha\rangle} f$, *using the nodes* $x_{m,k} = \frac{k}{m}$, *can be represented by means of divided differences as follows*

$$(1.14)\quad (L_m^{\langle\alpha\rangle} f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{j=o}^{m} (-1)^j \frac{x^{(j,-\alpha)}}{m^j} (D^j_{-\alpha} \varphi_m^{\langle\alpha\rangle}(o)) [o, \tfrac{1}{m}, \tfrac{2}{m}, \dots, \tfrac{j}{m}; f].$$

Remark 1.2. At this point we wish to note that if one considers the operator $L_m^{\langle\alpha;\beta,\gamma\rangle}$ obtained from (1.2) by taking the nodes $x_{m,k} = (k+\beta)/(m+\gamma)$, where $o \leq \beta \leq \gamma$, and $k = o(1)m$, then by means of the same method which we used for establishing (1.12) one finds the following representations

$$(1.12')\quad (L_m^{\langle\alpha;\beta,\gamma\rangle} f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{j=o}^{m} (-1)^j \frac{x^{(j,-\alpha)}}{j!} (D^j_{-\alpha} \varphi_m^{\langle\alpha\rangle}(o)) \Delta^j_{1/(m+\gamma)} f(\tfrac{\beta}{m+\gamma}) =$$

$$= (1/\varphi_m^{\langle\alpha\rangle}(o)) \sum_{j=o}^{m} (-1)^j \frac{x^{(j,-\alpha)}}{(m+\gamma)^j} (D^j_{-\alpha} \varphi_m^{\langle\alpha\rangle}(o)) [\tfrac{\beta}{m+\gamma}, \tfrac{\beta+1}{m+\gamma}, \dots, \tfrac{\beta+j}{m+\gamma}; f]$$

which in the special case $\alpha = o$ have been established in our paper [13].

3. We now state and prove an important approximation property of the operators introduced in this paper.

THEOREM 1.2. *Suppose that the parameter* α *is non-negative and depends on* m *in such a way that* $\alpha \to o$ *as* $m \to \infty$, *and that for each* $m = 1, 2, \dots$ *the polynomial* $\varphi_m^{\langle\alpha\rangle}$ *fulfills the conditions:*

i) $\varphi_m^{\langle\alpha\rangle}(o) > o$,

ii) $(-1)^k D^k_\alpha \varphi_m^{\langle\alpha\rangle}(x) \geq o, \quad x \in [o,1], \quad k = o(1)m$,

iii) $\displaystyle\lim_{m\to\infty} \frac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m\,\varphi_m^{\langle\alpha\rangle}(o)} = -1$ iv) $\displaystyle\lim_{m\to\infty} \frac{D^2_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m^2\varphi_m^{\langle\alpha\rangle}(o)} = 1.$

If $f \in C[o,1]$, *then the sequence* $(L_m^{\langle\alpha\rangle} f)$ *converges uniformly to* f *on* $[o,1]$.

Proof. Denoting by e_i the function defined, for all $x \in [o,1]$, by $e_i(x) = x^i$ $(i = o,1,2)$, we have

$$\Delta_{1/m} e_1(o) = \frac{1}{m}, \quad \Delta_{1/m} e_2(o) = \frac{1}{m^2}, \quad \Delta^2_{1/m} e_2(o) = \frac{2}{m^2}.$$

Consequently, by virtue of (1.12) we obtain at once

$$(1.15)\quad \begin{aligned} &(L_m^{\langle\alpha\rangle} e_o)(x) = 1, \quad (L_m^{\langle\alpha\rangle} e_1)(x) = \Big(-\frac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m\,\varphi_m^{\langle\alpha\rangle}(o)}\Big) x, \\ &(L_m^{\langle\alpha\rangle} e_2)(x) = \frac{D^2_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m^2\varphi_m^{\langle\alpha\rangle}(o)} x(x+\alpha) - \frac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m^2\varphi_m^{\langle\alpha\rangle}(o)} x. \end{aligned}$$

With these formulas in hand, it is readily seen that we have uniformly on $[o,1]$:

$$\lim_{m\to\infty} L_m^{\langle\alpha\rangle} e_i = e_i \qquad (i = o,1,2).$$

Now, since $(L_m^{\langle\alpha\rangle})$ is a sequence of positive linear operators, we may apply the well-known theorem of BOHMAN-KOROVKIN (see, e.g. [4] or [2]) and we conclude that the assertion of our theorem is valid.

§ 2. APPROXIMATION OF FUNCTIONS BY MEANS OF A CLASS OF GENERALIZED BERNSTEIN-TYPE SERIES

4. In order to obtain an extension of the positive linear polynomial operator, defined at (1.2), in the form of an infinite series, we consider a sequence of functions $(\varphi_m^{\langle\alpha\rangle})$, depending on a parameter α, which are analytic in a region D containing the disk $|z-a| \leq a$ $(a>0)$ and which can be expanded in a Newton's convergent series on D:

$$\varphi_m^{\langle\alpha\rangle}(t) = \sum_{k=0}^{\infty} \frac{(t-x)^{(k,\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x).$$

Assuming that $\varphi_m^{\langle\alpha\rangle}(0) \neq 0$, we may write the identity

$$(2.1) \qquad (1/\varphi_m^{\langle\alpha\rangle}(0)) \sum_{k=0}^{\infty} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = 1,$$

where $x \in I = [0,a]$.

Now to each function f defined on an interval J, such that $I \subseteq J$, we associate the operators $L_m^{\langle\alpha\rangle}$ defined by

$$(2.2) \qquad (L_m^{\langle\alpha\rangle}f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(0)) \sum_{k=0}^{\infty} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)\cdot f(x_{m,k}),$$

where $x \in I$ and $x_{m,k} \in J$, provided the series occuring in the right side of this formula is convergent.

By assuming that $\alpha \geq 0$ and that we have

$$(2.3) \qquad \varphi_m^{\langle\alpha\rangle}(0) > 0, \quad (-1)^k D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \geq 0 \quad (0 \leq x \leq a),$$

for each $k = 0,1,2,\ldots$ and $m = 1,2,\ldots$, one observes that the preceding linear operators are of positive type.

Obviously, if $\alpha > 0$ then we can write

$$(2.2') \qquad (L_m^{\langle\alpha\rangle}f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(0)) \sum_{k=0}^{\infty} \binom{-x/\alpha}{k} \Delta_\alpha^k \varphi_m^{\langle\alpha\rangle}(x)\cdot f(x_{m,k}).$$

The operators considered in this section include, as special cases, several important operators which generalize some well-known operators of Bernstein type, given in a polynomial form or in a power series form.

In the sequel we shall denote by p a given non-negative integer.

Examples. 1°. If we take

$$\varphi_m^{\langle\alpha\rangle}(x) = (1-x)^{(m+p,-\alpha)}, \quad a=1, \quad J=[0,1+\frac{p}{m}], \quad x_{m,k} = \frac{k+\beta}{m+\gamma},$$

then it follows that

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k (m+p)^{(k,1)} (1-x)^{(m+p-k,-\alpha)},$$

so that we obtain the operator $L_{m,p}^{\langle\alpha;\beta,\gamma\rangle}$ defined by

$$(2.4) \qquad (L_{m,p}^{\langle\alpha;\beta,\gamma\rangle}f)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m+p-k,-\alpha)}}{1^{(m+p,-\alpha)}} f\left(\frac{k+\beta}{m+\gamma}\right)$$

which in the case $p=\beta=\gamma=0$ reduces to our operator defined at (1.5).

If we take $\beta=\gamma=0$ and make $\alpha \to 0$, then we get the generalized Bernstein operator $B_{m,p}$ considered first by SCHURER [7] and then by SIKKEMA [9]:

$$(B_{m,p}f)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right),$$

which for $p=0$ becomes the classical mth Bernstein operator.

2°. By choosing

$$\varphi_m^{\langle\alpha\rangle}(x) = (1-\overline{m+p}\,\alpha)^{\frac{x}{\alpha}},$$

we have

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k (m+p)^k (1-\overline{m+p}\,\alpha)^{\frac{x}{\alpha}}$$

and in this case we get the operator $L_{m,p}^{\langle\alpha\rangle}$ defined by

$$(2.5)\qquad (L_{m,p}^{\langle\alpha\rangle} f)(x) = (1-\overline{m+p}\;\alpha)^{\frac{x}{\alpha}} \sum_{k=0}^{\infty} \frac{(m+p)^k x^{(k,-\alpha)}}{k!} f(x_{m,k}).$$

If we take $x_{m,k} = k/m$ and make $\alpha \to 0$ then it reduces to the generalized operator of Favard $(p = 0)$:

$$(L_{m,p} f)(x) = e^{-(m+p)x} \sum_{k=0}^{\infty} \frac{(m+p)^k x^k}{k!} f(\frac{k}{m}),$$

cosidered previously by SCHURER [7] and SIKKEMA [9].

3°. If we set

$$\varphi_m^{\langle\alpha\rangle}(x) = \frac{1}{(1+x)^{(m+p,-\alpha)}},$$

then it follows that

$$D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) = (-1)^k \frac{(m+p+k-1)^{(k,1)}}{(1+x)^{(m+p+k,-\alpha)}}$$

and we obtain the operator $L_{m,p}^{\langle\alpha\rangle}$ given by

$$(2.6)\qquad (L_{m,p}^{\langle\alpha\rangle} f)(x) = \sum_{k=0}^{\infty} \binom{m+p+k-1}{k} \frac{1^{(m+p,-\alpha)} x^{(k,-\alpha)}}{(1+x)^{(m+p+k,-\alpha)}} f(x_{m,k}),$$

which in the cases $p = 0$ and $x_{m,k} = k/m$ has been introduced and investigated in our paper [15], as a generalization of the well-known Baskakov operator $(\alpha \to 0)$.

5. Now we shall give representations of the operators defined at (2.2), when the nodes are equally spaced, by means of the finite differences and of the divided differences.

We have

THEOREM 2.1. *Denoting by* $L_m^{\langle\alpha;\beta,\gamma\rangle}$ *the operator defined at* (2.2) *in the case of the nodes* $x_{m,k} = (k+\beta)/(m+\gamma)$, *where* $0 \le \beta \le \gamma$, *we have the following representation*

$$(2.7)\quad (L_m^{\langle\alpha;\beta,\gamma\rangle} f)(x) = (1/\varphi_m^{\langle\alpha\rangle}(0)) \sum_{j=0}^{\infty} (-1)^j \frac{x^{(j,-\alpha)}}{j!} (D_{-\alpha}^j \varphi_m^{\langle\alpha\rangle}(0)) \cdot \Delta^j_{1/(m+\gamma)} f(\frac{\beta}{m+\gamma}),$$

whenever $x \in I$.

Proof. Following a method similar with that used for establishing formula (1.8), one can show that we have

$$(2.8)\qquad \begin{aligned} &\varphi_m^{\langle\alpha\rangle}(0) \cdot D_{-\alpha}^j (L_m^{\langle\alpha;\beta,\gamma\rangle} f)(x) = \\ &= (-1)^j \sum_{k=0}^{\infty} (-1)^k \frac{x^{(k,-\alpha)}}{k!} D_\alpha^k (D_{-\alpha}^j \varphi_m^{\langle\alpha\rangle}(x)) \cdot \Delta^j_{\frac{1}{m+\gamma}} f(\frac{\beta}{m+\gamma}). \end{aligned}$$

By expanding $L_m^{\langle\alpha;\beta,\gamma\rangle} f$ in a Newton's series on the nodes $0, -\alpha, -2\alpha, \ldots$ we have

$$(L_m^{\langle\alpha;\beta,\gamma\rangle} f)(x) = \sum_{j=0}^{\infty} \frac{x^{(j,-\alpha)}}{j!} D_{-\alpha}^j (L_m^{\langle\alpha;\beta,\gamma\rangle} f)(0).$$

Putting $x = 0$ in (2.8), we obtain

$$\varphi_m^{\langle\alpha\rangle}(0) D_{-\alpha}^j (L_m^{\langle\alpha;\beta,\gamma\rangle} f)(0) = (-1)^j D_{-\alpha}^j \varphi_m^{\langle\alpha\rangle}(0) \cdot \Delta^j_{\frac{1}{m+\gamma}} f(\frac{\beta}{m+\gamma})$$

and by insertion in the former expansion we arrive at the desired formula (2.7).

Evidently, as at (1.12') one can give a formula, similar with (2.7), by using divided differences.

The following two corollaries are immediate consequences of Theorem 2.1.

COROLLARY 2.1. *If* f *is a polynomial of degree* p, *then* $L_m^{\langle\alpha;\beta,\gamma\rangle} f$ *is a polynomial of degree not exceeding* p.

COROLLARY 2.2. *For any* $x \in I$ *we have*

$$(L_m^{\langle 0;\beta,\gamma\rangle} f)(x) = (1/\varphi_m(0)) \sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!} \varphi_m^{(j)}(0) \cdot \Delta^j_{\frac{1}{m+\gamma}} f(\frac{\beta}{m+\gamma}).$$

It should be remarked that for $\beta = \gamma = o$, $\varphi_m(o) = 1$ - the case of the Baskakov operators - this last formula has been given by LUPAŞ [5].

6. We are now prepared to present the main theorem of this section.

THEOREM 2.1. *Let α be a non-negative parameter depending on m such that $\alpha \to o$ as $m \to \infty$. Assuming that for each $m = 1,2,\dots$ the function $\varphi_m^{\langle\alpha\rangle}$ is analytic in a region D, containing the disk $|z-a| \leq a$ $(a > o)$, and satisfies the conditions*

i) $\varphi_m^{\langle\alpha\rangle}(o) > o,$

ii) $(-1)^k D_\alpha^k \varphi_m^{\langle\alpha\rangle}(x) \geq o, \quad x \in I = [o,a], \quad k = o,1,2,\dots$

iii) $\lim\limits_{m\to\infty} \dfrac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{m\,\varphi_m^{\langle\alpha\rangle}(o)} = -1,$

iv) $\lim\limits_{m\to\infty} \dfrac{D_{-\alpha}^2 \varphi_m^{\langle\alpha\rangle}(o)}{m^2\,\varphi_m^{\langle\alpha\rangle}(o)} = 1,$

then for any function f, defined on the interval J $(\supseteq I)$, continuous on $[o,a]$ and right-continuous at $x = a$, the sequence $(L_m^{\langle\alpha;\beta,\gamma\rangle}f)$ converges to f uniformly on $[o,a]$, β and γ being constants such that $o \leq \beta \leq \gamma$.

Proof. Since

$$\Delta_{\frac{1}{m+\gamma}} e_1\left(\frac{\beta}{m+\gamma}\right) = \frac{1}{m+\gamma}, \quad \Delta_{\frac{1}{m+\gamma}} e_2\left(\frac{\beta}{m+\gamma}\right) = \frac{1+2\beta}{(m+\gamma)^2}, \quad \Delta^2_{\frac{1}{m+\gamma}} e_2\left(\frac{\beta}{m+\gamma}\right) = \frac{2}{(m+\gamma)^2},$$

according to (2.7) we have

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_o)(x) = 1, \quad (L_m^{\langle\alpha;\beta,\gamma\rangle}e_1)(x) = \frac{\beta}{m+\gamma} - \frac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{(m+\gamma)\,\varphi_m^{\langle\alpha\rangle}(o)}\,x,$$

(2.9)

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_2)(x) = \left(\frac{\beta}{m+\gamma}\right)^2 - \frac{1+2\beta}{(m+\gamma)^2}\,\frac{D_{-\alpha}\varphi_m^{\langle\alpha\rangle}(o)}{\varphi_m^{\langle\alpha\rangle}(o)}\,x + \frac{1}{(m+\gamma)^2}\,\frac{D_{-\alpha}^2\varphi_m^{\langle\alpha\rangle}(o)}{\varphi_m^{\langle\alpha\rangle}(o)}\,x(x+\alpha).$$

Consequently, we have uniformly on $[o,a]$:

$$\lim_{m\to\infty} L_m^{\langle\alpha;\beta,\gamma\rangle} e_i = e_i \qquad (i = o,1,2)$$

and by virtue of the Bohman-Korovkin theorem we can conclude that the assertion of Theorem 2.1. is true.

7. We shall now present some convergence theorems concerning the special classes of linear positive operators considered in § 2.

1^o. In the case of the operators defined at (2.4), by taking $p = o$ we have:

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_o)(x) = 1, \quad (L_m^{\langle\alpha;\beta,\gamma\rangle}e_1)(x) = x + \frac{\beta-\gamma x}{m+\gamma},$$

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_2)(x) = \frac{1}{(m+\gamma)^2}\left[\beta^2 + (1+2\beta)mx + \frac{m(m-1)}{1+\alpha}\,x(x+\alpha)\right].$$

Therefore, we can state

THEOREM 2.2 *If $f \in C[o,1]$ and $o \leq \alpha = \alpha_m \to o$ as $m \to \infty$, then the sequence $(L_m^{\langle\alpha;\beta,\gamma\rangle}f)$ converges uniformly to f on $[o,1]$.*

2^o. Let $x_{m,k} = (k+\beta)/(m+\gamma)$ and $p = o$. It is easily checked that for the corresponding operator given at (2.5) we find:

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_o)(x) = 1, \quad (L_m^{\langle\alpha;\beta,\gamma\rangle}e_1)(x) = \frac{\beta}{m+\gamma} + \frac{mx}{(1-m\alpha)(m+\gamma)},$$

$$(L_m^{\langle\alpha;\beta,\gamma\rangle}e_2)(x) = \frac{1}{(m+\gamma)^2}\left[\beta^2 + \frac{m(1+2\beta)x}{1-m\alpha} + \frac{m^2 x(x+\alpha)}{(1-m\alpha)^2}\right].$$

Consequently, we can state the following result

THEOREM 2.3. *Let f be bounded on $[o,\infty)$, continuous on $[o,a]$, where $a > o$, and right-continuous at $x = a$. If $o \leq \alpha = \alpha_m < m^{-1}$, then the sequence $(L_m^{\langle\alpha;\beta,\gamma\rangle}f)$ converges uniformly to f on $[o,a]$.*

3^o. Finally, we consider the case of the operator (2.6), where we take $x_{m,k} = (k+\beta)/(m+\gamma)$. According to (2.9) we have

$$(L_{m,p}^{\langle\alpha;\beta,\gamma\rangle} e_o)(x) = 1, \quad (L_{m,p}^{\langle\alpha;\beta,\gamma\rangle} e_1)(x) = \frac{\beta}{m+\gamma} + \frac{(m+p)x}{(1-\alpha)(m+\gamma)},$$

$$(L_{m,p}^{\langle\alpha;\beta,\gamma\rangle} e_2)(x) = \frac{1}{(m+\gamma)^2}\left[\beta^2 + \frac{(1+2\beta)(m+p)x}{1-\alpha} + \frac{(m+p)(m+p+1)x(x+\alpha)}{(1-\alpha)(1-2\alpha)}\right].$$

Thus, we are in position to formulate

THEOREM 2.4. *Denoting by f a function bounded on $[o,\infty)$, continuous on $[o,a]$, where $a > o$, and right-continuous at $x = a$, if $o \leq \alpha = \alpha_m \to o$ as $m \to \infty$, then the sequence $(L_{m,p}^{\langle\alpha;\beta,\gamma\rangle} f)$, where p is a non-negative integer, converges uniformly to f on $[o,a]$.*

8. We end this paper mentioning that according to the results of MAMEDOV [6] and of SHISHA and MOND [8], one can give estimates for the approximation of a continuous function f on $[o,a]$ by means of the associated operator, defined at (2.2), of the following form:

$$\| f - L_m^{\langle\alpha;\beta,\gamma\rangle} f \| \leq 2\omega\,(f;\, \delta_m^{\langle\alpha\rangle}),$$

where $\|.\|$ stands for the *sup norm* over $[o,a]$, ω is the modulus of continuity, while

$$\delta_m^{\langle\alpha\rangle} = \| L_m^{\langle\alpha;\beta,\gamma\rangle} \psi_x \|^{1/2},$$

ψ_x being a function defined as follows: $\psi_x(t) = (t-x)^2$ for all t in the basic interval $[o,a]$, x having any fixed value in $[o,a]$; the operator involved is to be performed with respect to t.

By making use of the results given at (2.9) one can find effective values for $\delta_m^{\langle\alpha\rangle}$.

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