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A NEW GENERALIZATION OF THE MEYER-KÖNIG AND ZELLER OPERATORS

BY

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1. In 1970 we have introduced and studied, în a paper published in this journal [12], a new sequence of positive linear operators $(W_m^{[\alpha]})_{m=1}^{\infty}$, of Meyer-König and Zeller type, defined by

$$(W_m^{[\alpha]} f)\ (x) = \sum_{k=0}^{\infty} \binom{m+k}{k} \frac{x(x+\alpha)\ldots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\ldots(1-x+m\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{m+k}\alpha)} f\left(\frac{k}{m+k}\right),$$

where $0 \leqq x \leqq 1$, f is a function defined and bounded on $[0, 1]$, while α is a non-negative parameter which may depend only on m.

Obviously, for $\alpha=0$ these operators reduce to the well-known [4] Meyer-König and Zeller operators M_m, given by

$$(M_m f)\ (x) = \sum_{k=0}^{\infty} \binom{m+k}{k} x^k (1-x)^{m+1} f\left(\frac{k}{m+k}\right), \tag{1}$$

which have been investigated in many recent papers (see, e.g., [3] – [9], [12] – [13]).

Recently, Swetits and Wood [14] constructed these operators by using a variation of the Markov-Pólya urn scheme, employing a procedure presented in our earlier paper [11].

2. In the present paper we give a new generalization of the operators of Meyer-König and Zeller, associating to each function f, continuous on $[0,1]$, the operators $U_m^{[\alpha]}$ $(m=1, 2, \ldots)$ defined by

$$(U_m^{[\alpha]} f)\ (x) = \sum_{k=0}^{\infty} u_{m,\,k}^{[\alpha]}(x) f\left(\frac{k}{m+k}\right), \tag{2}$$

where $0 \leqq x < 1$ and

$$u_{m,\,k}^{[\alpha]}(x) = \binom{m+k}{k} (1-x)^{m+1} \prod_{i=0}^{m} (1+i\alpha) \cdot \prod_{j=0}^{k-1} [x+j\alpha(1-x)] \Big/ \prod_{\nu=0}^{m+k} [1+\nu\alpha(1-x)], \tag{3}$$

α being a non-negative parameter which might depend on m.

It is evidently that $(U_m^{[0]} f)(x) = (M_m f)(x)$ for any $x \in [0, 1)$.
We note that we may write

$$u_{m+k}^{[\alpha]}(x) = \binom{m+k}{k} \prod_{i=0}^{m} (1+i\alpha) \cdot \prod_{j=0}^{k-1}\left(\frac{x}{1-x}+j\alpha\right) \Big/ \prod_{\nu=0}^{m+k}\left(\frac{1}{1-x}+\nu\alpha\right),$$

which permits to see that the operators $U_m^{[\alpha]}$ can actually be obtained by st ting from the operators $V_m^{[\alpha]}$ defined by

$$(V_m^{[\alpha]} f)(y) = \sum_{k=0}^{\infty} \binom{m+k-1}{k}\left[\prod_{i=0}^{m-1}(1+i\alpha)\cdot\prod_{j=0}^{k-1}(y+j\alpha)\Big/\prod_{\nu=0}^{m+k-1}(1+y+\nu\alpha)\right] f\left(\frac{k}{m}\right)$$

introduced and investigated in our paper [12] as a generalization of the we known Baskakov's operators, performing the change of variable $y=\frac{x}{1-x}$ then replacing m by $m+1$ and taking, as Cheney-Sharma [1] have done in th case of the Meyer-König and Zeller original operators, in place of the co responding nodes $\frac{k}{m+k+1}$, the nodes $\frac{k}{m+k}$.

It should be noticed that the operators $W_m^{[\alpha]}$ and $U_m^{[\alpha]}$ generalize the operators M_m in the same sens as the operators $P_m^{[\alpha]}$ introduced in [10] generalize the operators B_m of Bernstein.

3. For proving a convergence theorem corresponding to the sequence of the operators $(U_m^{[\alpha]})$, it is helpful to establish first two lemmas.

Lemma 1. *If* $\alpha>0$ *and* $0<x<1$ *then we have the following representation*

$$(U_m^{[\alpha]} f)(x) = \frac{1}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha(1-x)}\right)} \int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^{\frac{x}{\alpha(1-x)}-1}(Q_m f)(t)\,dt, \tag{4}$$

where

$$(Q_m f)(t) = \sum_{k=0}^{\infty}\binom{m+k}{k} t^{m+1}(1-t)^k f\left(\frac{k}{m+k}\right) \tag{5}$$

and B is the Euler beta function.

Proof. It is easily verified that

$$u_{m,k}^{[\alpha]}(x) = \frac{B\left(\frac{1}{\alpha}+m+1, \frac{x}{\alpha(1-x)}+k\right)}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha(1-x)}\right)}.$$

Consequently we can write successively

$$B\left(\frac{1}{\alpha}, \frac{x}{\alpha(1-x)}\right)(U_m^{[\alpha]} f)(x) =$$

$$= \sum_{k=0}^{\infty}\binom{m+k}{k} B\left(\frac{1}{\alpha}+m+1, \frac{x}{\alpha(1-x)}+k\right) f\left(\frac{k}{m+k}\right) =$$

$$= \sum_{k=0}^{\infty}\binom{m+k}{k}\left[\int_0^1 t^{\frac{1}{\alpha}+m}(1-t)^{\frac{x}{\alpha(1-x)}+k-1}dt\right] f\left(\frac{k}{m+k}\right) =$$

$$= \int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^{\frac{x}{\alpha(1-x)}-1}\left[\sum_{k=0}^{\infty}\binom{m+k}{k} t^{m+1}(1-t)^k f\left(\frac{k}{k+m}\right)\right]dt$$

since $Q_m f$ is bounded on $[0,1]$ and is represented by a uniformly convergent series on any interval $[c,d]$, where $0<c<d<1$, so that the interchange of the order of integration and summation is allowed. Thus the Lemma 1 is proved.

When $x=0$ it can be directly observed that we have: $(U_m^{[\alpha]} f)(0)=f(0)$.

Lemma 2. *Let* $e_i(x)=x^i$ $(i=0, 1, 2)$ *for* $x\in[0,1)$. *The following formulas*

$$(U_m^{[\alpha]} e_0)(x)=1, \quad (U_m^{[\alpha]} e_1)(x)=x, \quad (U_m^{[\alpha]} e_2)(x)= \tag{6}$$

$$= \frac{x[x+\alpha(1-x)]}{1+\alpha(1-x)} + \frac{(1+\alpha)x(1-x)^2}{m[1+\alpha(1-x)][1+2\alpha(1-x)]} + 0\left(\frac{1}{m}\right) \quad (m\to\infty)$$

hold.

Proof. First, we assume that $\alpha>0$ and $0<x<1$.

Since by using the expansion in the binomial series

$$(1-y)^{-\gamma} = \sum_{k=0}^{\infty}\binom{\gamma+k-1}{k} y^k \quad (|y|<1),$$

we obtain

$$(Q_m e_0)(t) = t^{m+1}\sum_{k=0}^{\infty}\binom{m+k}{k}(1-t)^k = t^{m+1}\cdot t^{-(m+1)} = 1, \tag{7}$$

it follows that we have

$$(U_m^{[\alpha]} e_0)(x) = B\left(\frac{1}{\alpha}, \frac{x}{\alpha(1-x)}\right)\Big/ B\left(\frac{1}{\alpha}, \frac{x}{\alpha(1-x)}\right) = 1.$$

If we proceed in the same manner we can write

$$(Q_m e_1)(t) = t^{m+1}\sum_{k=1}^{\infty}\binom{m+k-1}{k-1}(1-t)^k = t^{m+1}\sum_{j=0}^{\infty}\binom{m+j}{j}(1-t)^{j+1} =$$

$$= t^{m+1}\cdot(1-t)\,t^{-(m+1)} = 1-t,$$

because

$$\frac{k}{m+k}\binom{m+k}{k}=\binom{m+k-1}{k-1}.$$

Consequently we obtain

$$(U_m^{[\alpha]}e_1)(x)=B\left(\frac{1}{\alpha},\frac{x}{\alpha(1-x)}+1\right)\Big/B\left(\frac{1}{\alpha},\frac{x}{\alpha(1-x)}\right)=x.$$

Going on to e_2 we get

$$(Q_m e_2)(t)=\sum_{k=1}^{\infty}\binom{m+k}{k}t^{m+1}(1-t)^k\left(\frac{k}{m+k}\right)^2. \qquad (8)$$

Since

$$\frac{k^2}{(m+k)^2}\binom{m+k}{k}=\frac{k(k-1)+k}{(m+k)^2}\binom{m+k}{k}=\frac{m+k-1}{m+k}\binom{m+k-2}{k-2}+$$

$$+\frac{k}{(m+k)^2}\binom{m+k}{k}=\binom{m+k-2}{k-2}-\frac{1}{m+k}\binom{m+k-2}{k-2}+\frac{1}{m+k}\binom{m+k-1}{k-1},$$

by substituting into (8) and then by making obvious changes of indices of summations, we get

$$(Q_m e_2)(t)=\sum_{j=0}^{\infty}\left[\binom{m+j}{j}-\frac{1}{m+j+2}\binom{m+j}{j}\right]t^{m+1}(1-t)^{j+2}+$$

$$+\sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m+j}{j}t^{m+1}(1-t)^{j+1}.$$

By using the relation

$$\frac{1}{m+j+p}\binom{m+j}{j}=\frac{1}{m}\binom{m+j-1}{j}-\frac{p}{m(m+j+p)}\binom{m+j-1}{j}$$

for $p=1,\ 2$ and the identity (7), we obtain

$$(Q_m e_2)(t)=(1-t)^2+\frac{t^2(1-t)}{m}+\frac{2}{m}\sum_{j=0}^{\infty}\frac{1}{m+j+2}\binom{m+j-1}{j}t^{m+1}(1-t)^{j+2}-$$

$$-\frac{1}{m}\sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m+j-1}{j}t^{m+1}(1-t)^{j+1}.$$

Consequently, according to (4) we have

$$(U_m^{[\alpha]}e_2)(x)=\Big[B\left(\frac{1}{\alpha},\frac{x}{\alpha(1-x)}+2\right)+\frac{1}{m}B\left(\frac{1}{\alpha}+2,\frac{x}{\alpha(1-x)}+1\right)+$$

$$+\frac{2}{m}\sum_{j=0}^{\infty}\frac{1}{m+j+2}\binom{m+j-1}{j}B\left(\frac{1}{\alpha}+m+1,\frac{x}{\alpha(1-x)}+j+2\right)-$$

$$-\sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m+j-1}{j}B\left(\frac{1}{\alpha}+m+1,\frac{x}{\alpha(1-x)}+j+1\right)\Big]\Big/B\left(\frac{1}{\alpha},\frac{x}{\alpha(1-x)}\right)=$$

$$=\frac{x[x+\alpha(1-x)]}{1+\alpha(1-x)}+\frac{(1+\alpha)x(1-x)^2}{m[1+\alpha(1-x)][1+2\alpha(1-x)]}+\rho_m^{[\alpha]}(x),$$

where

$$\rho_m^{[\alpha]}(x)=\frac{2}{m}\sum_{j=0}^{\infty}\frac{1}{m+j+2}\binom{m+j-1}{j}g_{m,j+1}^{[\alpha]}(x)-$$

$$-\frac{1}{m}\sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m+j-1}{j}g_{m,j}^{[\alpha]}(x),$$

with

$$g_{m,j}^{[\alpha]}(x)=\prod_{i=1}^{m}(1+i\alpha)\cdot\prod_{\nu=0}^{j}[x+\nu\alpha(1-x)]\cdot(1-x)^{m+1}\Big/\prod_{\mu=1}^{j}[1+\mu\alpha(1-x)].$$

Because $\rho_m^{[\alpha]}(x)=0\left(\frac{1}{m}\right)$ as $m\to\infty$, we obtain the required asymptotic formula for $(U_m^{[\alpha]}e_2)(x)$.

Since $(U_m^{[\alpha]}f)(0)=f(0)$, it is evident that formulas (6) are also true for $x=0$. On the other hand, if we take into account the fact that $(U_m^{[0]}f)(x)=(M_m f)(x)$ for $x\in[0,\ 1)$ and the corresponding relations established by Müller [5], we can see that formulas (6) also hold if $\alpha=0$. This completes the proof of Lemma 2.

3. We are now able to state and prove a convergence theorem for the sequence of the linear positive operators $U_m^{[\alpha]}$ mapping the linear space $C[0,\ 1]$ of continuous real-valued functions on the compact unit interval $[0,\ 1]$ into itself.

Theorem 1. *Let $f\in C[0,\ 1]$. If $0\leqq\alpha=\alpha_m\to 0$ as $m\to\infty$, then we have*

$$\lim_{m\to\infty}U_m^{[\alpha]}f=f$$

uniformly on each interval of the form $[0,\ a]$, where $0<a<1$.

Proof. We may consider $\alpha>0$, because for $\alpha=0$ this theorem has been proved by Meyer-König and Zeller [4].

According to Lemma 2, we have uniformly on [0, a]:

$$\lim_{m\to\infty} U_m^{[\alpha]} e_j = e_j \quad (j=0,\ 1,\ 2).$$

Appealing to the well-known Bohman-Korovkin theorem (see, e. g.: [2]) we can conclude that the assertion of our theorem is true.

5. By using a method presented in § 3 of our previous paper [10] one can evaluate the order of approximation of $f \in C[0,1]$ by means of $U_m^{[\alpha]} f$.

First we state and prove

Lemma 3. *The following inequality*

$$\frac{x\,[x+\alpha(1-x)]}{1+\alpha(1-x)} \leqq (U_m^{[\alpha]} e_2)(x) \leqq \frac{x\,[x+\alpha(1-x)]}{1+\alpha(1-x)} + \frac{x(1-x)}{m\,[1+\alpha(1-x)]} \tag{9}$$

holds for all $x \in [0,1]$.

Proof. If in (8) we use the relation

$$\frac{k^2}{(m+k)^2}\binom{m+k}{k} = \binom{m+k-2}{k-2} + \frac{1}{m+k}\binom{m+k-2}{k-1},$$

which can be easily verified, we obtain

$$(Q_m e_2) = \sum_{k=2}^{\infty}\binom{m+k-2}{k-2} t^{m+1}(1-t)^k + \sum_{k=1}^{\infty}\frac{1}{m+k}\binom{m+k-2}{k-1} t^{m+1}(1-t)^k =$$

$$= \sum_{j=0}^{\infty}\binom{m+j}{j} t^{m+1}(1-t)^{j+2} + \sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m-1+j}{j} t^{m+1}(1-t)^{j+1} =$$

$$= (1-t)^2 + t(1-t)\sum_{j=0}^{\infty}\frac{1}{m+j+1}\binom{m-1+j}{j} t^m (1-t)^j.$$

Consequently we can write relation

$$(1-t)^2 \leqq (Q_m e_2)(t) \leqq (1-t)^2 + \frac{t(1-t)}{m}.$$

Using this and Lemma 1, we get just the relation (9), which is also true if $x=0$ and if $\alpha=0$.

Theorem 2. *If* $f \in C[0,1]$ *and* $\alpha \geq 0$, *then we have*

$$\| f - U_m^{[\alpha]} f \| \leqq \frac{2+\sqrt{\alpha+1}}{1+\sqrt{\alpha+1}}\,\omega_f\left(\sqrt{\frac{1+\alpha m}{m}}\right), \tag{10}$$

where ω_f *is the modulus of continuity of* f *and* $\|\cdot\|$ *stands for the uniform norm over* $[0,a]$ $(0<a<1)$.

Proof. According to the first relation from (6) and the fact that the polynomials defined at (3) are $\geqq 0$ on [0,1), we can write

$$| f(x) - (U_m^{[\alpha]} f)(x) | \leqq \sum_{k=0}^{\infty} u_{m,k}^{[\alpha]}(x) \left| f(x) - f\left(\frac{k}{m+k}\right)\right|.$$

By using the following properties of the modulus of continuity

$$| f(x'') - f(x') | \leqq \omega_f(|x''-x'|),\quad \omega_f(\lambda\delta) \leqq (\lambda+1)\,\omega_f(\delta) \quad (\lambda,\delta>0),$$

we obtain

$$\left| f(x) - f\left(\frac{k}{m+k}\right)\right| \leqq \omega_f\left(\left|x-\frac{k}{m+k}\right|\right) \leqq \omega_f\left(\frac{1}{\delta}\left|x-\frac{k}{m+k}\right|\delta\right) \leqq$$

$$\leqq \left(1+\frac{1}{\delta}\left|x-\frac{k}{m+k}\right|\right)\omega_f(\delta).$$

Hence we can write

$$| f(x) - (U_m^{[\alpha]} f)(x) | \leqq \left[1+\frac{1}{\delta}(U_m^{[\alpha]}\psi_x)(x)\right]\omega_f(\delta), \tag{11}$$

where $\psi_x(t) = |x-t|$ for all t in the basic interval $[0,a]$, x being any fixed point in $[0,a]$.

According to the Cauchy-Schwarz inequality we have

$$(U_m^{[\alpha]}\psi_x)(x) \leqq [(U_m^{[\alpha]}\psi_x^2)(x)]^{1/2}$$

and by (6) we obtain

$$(U_m^{[\alpha]}\psi_x^2)(x) = (U_m^{[\alpha]} e_2)(x) - x^2.$$

By Lemma 3 and a straightforward calculation we find

$$\frac{\alpha x(1-x)^2}{1+\alpha(1-x)} \leqq (U_m^{[\alpha]}\psi_x^2)(x) \leqq \frac{\alpha x(1-x)^2}{1+\alpha(1-x)} + \frac{x(1-x)}{m[1+\alpha(1-x)]}.$$

It follows that

$$\| U_m^{[\alpha]}\psi_x \| \leqq \frac{1}{1+\sqrt{\alpha+1}}\sqrt{\frac{1+\alpha m}{m}}.$$

If we return to (11) we can write

$$\| f - U_m^{[\alpha]} f \| \leqq \left[1+\frac{1}{\delta(1+\sqrt{\alpha-1})}\sqrt{\frac{1+\alpha m}{m}}\right]\omega(\delta)$$

and by taking

$$\delta = \sqrt{\frac{1+\alpha m}{m}}$$

we readily arrive at the required inequality (10).

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O NOUĂ GENERALIZARE A OPERATORULUI LUI MEYER-KÖNIG ŞI ZELLER

REZUMAT

În anul 1970 autorul a introdus şi studiat, într-o lucrare apărută în această revistă, un operator liniar pozitiv care generalizează binecunoscutul operator al lui Meyer-König şi Zeller. În această lucrare se prezintă o nouă generalizare a acestui operator şi se studiază proprietăţile sale de aproximare a funcţiilor continue pe intervalul [0,1].