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ON THE APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY MEANS OF A CLASS OF LINEAR OPERATORS

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Summary. In this paper we develop a theory concerning the approximation of functions of two variables $f\in C(D)$, where $D=\{(x,y):x\geqq 0,\ y\geqq 0,\ x+y\leqq 1\}$, by means of the linear positive operator $P_m^{[a]}$ defined as follows

$$P_m^{[a]}f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}w_{m,i,j}^{[a]}(x,y)\,f\,(i/m,\ j/m),$$

where

$$w_{m,i,j}^{[a]}(x,y)=1^{(-m,-a)}\binom{m}{i}\binom{m-i}{j}x^{(i,-a)}y^{(j,-a)}(1-x-y)^{(m-i-j,-a)},$$

a being a parameter which may depend only on m, while by $u^{[n,h]}$ one denotes the generalized factorial of degree n and increment h of u.

1. In this paper we consider a sequence of positive linear operators $(P_m^{[a]})$, depending on a real parameter a, defined on the vector space $C(D)$ of functions f, of two variables, continuous on the isosceles right triangle $D:\{(x,y):x\geqq 0,\ y\geqq 0,\ x+y\leqq 1\}$, and we give some results concerning the approximation of functions f by means of $P_m^{[a]}f$.

Preliminary we point out that $(P_m^{[a]}f)(x,y)$ can be considered as originating in an identity derived from the Vandermonde formula for the factorial power and then we present a probabilistic method for obtaining it.

The results of the present paper represent extensions to two variables of those given in [9]. It should be mentioned that in our previous paper [11] we have considered the operators $P_m^{[a]}$ in the case $a\geqq 0$ as a generalization to the two-dimensional case of the polynomial operator of Bernstein type which we have introduced and investigated in [7]. In [8] and [10] we have discussed some probabilistic methods for constructing and investigating positive linear operators.

2. We denote by $u^{(n,h)}$ the factorial power of order n (: natural number) and increment h (: real number) of u, that is:

$$u^{(n,h)}=u(u-h)\cdots(u-\overline{n-1}h),$$

where $\overline{n-1}\, h$ represents the product $(n-1)h$.

If $h=1$, for brevity, we shall use the notation $u^{(n)}$ instead of $u^{(n,1)}$. When $n=0$ and $u \neq 0$ we set $u^{(0,h)}=1$. Evidently for $h=0$ we have $u^{(n,0)}=u^n$.

We shall make use of the following notations

$$u^{(-n,h)}=\frac{1}{u^{(n,h)}},\ u^{(-n,-h)}=\frac{1}{u^{(n,-h)}}=\frac{1}{(u+\overline{n-1}h)^{(n,h)}},$$

provided that the denominators are different from zero.

It is easily verified that if n and j are natural numbers such that $n>j$, then we have

$$u^{(n,j)}=u^{(j,n)}(u-jh)^{(n-j,h)}. \tag{1}$$

3. We now wish to point out that in the same manner as the Bernstein polynomial $B_m f$, defined by

$$(B_m f)(x,y)=\sum_{0\leq i+j\leq m}\frac{m!}{i!\,j!\,(m-i-j)!}x^i y^j(1-x-y)^{m-i-j}f\left(\frac{i}{m},\frac{j}{m}\right) \tag{2}$$

originates in the identity

$$1=(x+y+1-x-y)^m=\sum_{0\leq i+j\leq m}\frac{m!}{i!j!(m-i-j)!}x^i y^j(1-x-y)^{m-i-j},$$

so if we consider the Vandermonde formula

$$(u+v+w)^{(m,h)}=\sum_{\substack{i,j,k=0\\(i+j+k=m)}}^{m}\frac{m!}{i!j!k!}u^{(i,h)}v^{(j,h)}w^{(k,h)}$$

and set $u=x$, $v=y$, $w=1-x-y$, $h=-\alpha$, we obtain the identity

$$1^{(m,-\alpha)}=\sum_{0\leq i+j\leq m}\frac{m!}{i!j!(m-i-j)!}x^{(i,-\alpha)}y^{(j,-\alpha)}(1-x-y)^{(m-i-j,-\alpha)}$$

which gives rise to the polynomial $P_m^{[\alpha]}f$ defined by

$$(P_m^{[\alpha]}f)(x,y)=1^{(-m,-\alpha)}\sum_{0\leq i+j\leq m}\frac{m!}{i!j!(m-i-j)!}x^{(i,-\alpha)}y^{(j,-\alpha)} \tag{3}$$

$$\times(1-x-y)^{(m-i-j,-\alpha)}f\left(\frac{i}{m},\frac{j}{m}\right).$$

Throughout this paper we assume that the real parameter α may depend only on the natural number m so that

$$1^{(m,-\alpha)}=(1+\alpha)(1+2\alpha)\dots(1+\overline{m-1}\alpha)\neq 0.$$

It is obvious that at (3) we have a polynomial of degree m with respect to x and y, which can be written in the form

$$(P_m^{[\alpha]}f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}w_{m,i,j}^{[\alpha]}(x,y)f\left(\frac{i}{m},\frac{j}{m}\right), \tag{3'}$$

where

$$(3'') \qquad w^{[\alpha]}_{m,i,j}(x,y)=\binom{m}{i}\binom{m-i}{j}\frac{x^{(i,-\alpha)}y^{(j,-\alpha)}(1-x-y)^{(m-i-j,-\alpha)}}{1^{(m,-\alpha)}}.$$

4. As we have shown in [11], if $\alpha>0$, $x>0$, $y>0$ and $x+y<1$, then we can represent the operator $P^{[\alpha]}_m$ by means of the Bernstein operator B_m in the following form

$$(P^{[\alpha]}_m f)(x,y)$$

$$=\frac{1}{B\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{1-x-y}{\alpha}\right)}\iint_D u^{\frac{x}{\alpha}-1}v^{\frac{y}{\alpha}-1}(1-u-v)^{\frac{1-x-y}{\alpha}-1}(B_m f)(u,v)\,du\,dv,$$

where by $B(a,b,c)$ one denotes the Dirichlet double integral, that is

$$B(a,b,c)=\iint_D u^{a-1}v^{b-1}(1-u-v)^{c-1}\,du\,dv.$$

5. In accordance with our previous investigation [10] of the operator $P^{[\alpha]}_m$ we should notice that one can arrive at this operator by making use of the Markov-Pólya bivariate probability distribution.

A two-dimensional discrete random variable (X,Y) is said to have a Markov-Pólya distribution if its probabiliy function: $P(X=i,\ Y=j)$, where $i=0,1,\ldots,m$ and $j=0,1,\ldots,m-i$, is given by

$$(4) \qquad P(X=i,Y=j)=\frac{m^{(i+j)}}{i!\,j!}\,\frac{p^{(i,-\alpha)}q^{(j,-\alpha)}r^{(m-i-j,-\alpha)}}{1^{(m,-\alpha)}},$$

where p,q,r and α are any real numbers such that $p>0, q>0, r>0$ $p+q+r=1$, $-m\alpha\leq\min\{p,q,r\}$.

One can arrive at this formula when p,q and r are rational numbers by considering an urn model. This can be briefly explained as follows. An urn contains A white, B black and C red balls. One draws one ball at random and then it is replaced and moreover a balls of color drawn are added. This procedure is repeated m times. If we assume that the random vector (X,Y) takes on the values (i,j), where $i=0,1,\ldots,m$ and $j=0,1,\ldots,m-i$, if during m trials one obtains exactly i times a white ball, j times a black ball and $m-i-j$ times a red ball, then

$$P(X=i,Y=j)=\frac{m^{(i+j)}}{i!\,j!}\,\frac{A^{(i,-a)}B^{(j,-a)}C^{(m-i-j,-a)}}{(A+B+C)^{(m,-a)}}.$$

Now introducing the notations: $A+B+C=N$, $A/N=p$, $B/N=q$, $C/N=r$, $a/N=\alpha$, then the foregoing probability is given by a formula of the form (4).

If f is a real-valued function of two variables such that the mean value of $f(X/m,\ Y/m)$ exists for $m=1,2,\ldots$, then it is given just by $(P^{[\alpha]}_m f)\ (p,q)$.

It should be observed that the integer a of balls added at each trial may be chosen negative, when one can give the following interpretation:

after each drawing we do not replace the ball drawn and moreover we shall eliminate from the urn $-(a+1)$ balls of the same colour as the ball just drawn.

In order that we can perform all the m trials when a is a negative integer we should assume that

$$A+ma\geqq 0,\ B+ma\geqq 0,\ C+ma\geqq 0, \tag{5}$$

from which we find the restrictions: $p+ma\geqq 0$, $q+ma\geqq 0$, $r+ma=1-p-q+ma\geqq 0$. It is clear that these inequalities are satisfied if $a\geqq 0$, but when a is negative we should assume that $-ma\leqq 1/3$. In fact when $-ma=1/3$ the foregoing inequalities are satisfied only for $p=q=1/3$.

Consequently if we assume that $a=a_m$ is a negative number depending on m so that $-ma_m\leqq\varepsilon$, where $0<\varepsilon<1/3$, then we can define the operator $P_m^{[a]}$ by formulas (3) or (3′) and it is of positive type on the triangle D_ε whose vertices are the points

$$(\varepsilon,\varepsilon),\ (1-\varepsilon,\varepsilon),\ (\varepsilon,1-\varepsilon). \tag{6}$$

It is obvious that when $a=a_m\geqq 0$ then $P_m^{[a]}$ is of positive type over $D_0=D$.

6. Now we state and prove

Theorem 1. The polynomial defined by (3′)—(3″) can be represented by the following expansion

$$(P_m^{[a]}f)(x,y)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}\binom{m}{r}\binom{m-r}{s}\frac{x^{(r,-a)}y^{(s,-a)}}{1^{(r+s,-a)}}\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0), \tag{7}$$

where we have

$$\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0)=\sum_{\nu=0}^{r}\sum_{\mu=0}^{r}(-1)^{\nu+\mu}\binom{r}{\nu}\binom{s}{\mu}f\left(\frac{r-\nu}{m},\frac{s-\mu}{m}\right),$$

i. e., the finite difference of order (r,s) with the steps $h=k=1/m$ and the starting point (0,0) of the function f.

P r o o f. Let us consider the Newton-Biermann interpolating polynomial (see [4] and [6]) corresponding to the function f and the nodes $(i/m,\ j/m)$, where $i=0,1,\ldots,m$ and $j=0,1,\ldots,m-i$:

$$(N_mf)(t,\tau)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}\frac{(mt)^{(r)}(m\tau)^{(s)}}{r!s!}\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0).$$

Performing the changes of variables $mt=u$, $m\tau=v$ we obtain

$$(N_mf)\left(\frac{u}{m},\frac{v}{m}\right)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}\frac{u^{(r)}v^{(s)}}{r!s!}\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0).$$

From this we readily arrive at the equation

$$\sum_{0\leqq i+j\leqq m}w^{[a]}_{m,i,j}(x,y)(N_mf)\left(\frac{i}{m},\frac{j}{m}\right)=\sum_{0\leqq r+s\leqq m}\frac{M_{(r,s)}}{r!s!}\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0),$$

where

$$M_{(r,s)}=\sum_{0\le i+j\le m} i^{(r)}j^{(s)}w^{[\alpha]}_{m,i,j}(x,y).$$

According to the interpolating properties of the polynomial $N_m f$ we have $(N_m f)(i/m, j/m)=f(i/m, j/m)$ and we conclude that

(8) $$(P^{[\alpha]}_m f)(x,y)=\sum_{0\le r+s\le m}\frac{M_{(r,s)}}{r!s!}\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0).$$

We will now proceed to find a convenient formula for $M_{(r,s)}$. Because of the fact that $k^{(n)}=0$ if $k=0,1,\ldots, n-1$, we can write

(9) $$1^{(m,-\alpha)}M_{(r,s)}=\sum_{i=r}^{m}\sum_{j=s}^{m-i}i^{(r)}j^{(s)}\binom{m}{i}\binom{m-i}{j}x^{(i,-\alpha)}y^{(j,-\alpha)}(1-x-y)^{(m-i-j,-\alpha)}.$$

Refering to the formula

(10) $$k^{(q)}\binom{n}{k}=n^{(q)}\binom{n-q}{k-q}$$

we have

(11) $$S=\sum_{j=s}^{m-i}j^{(s)}\binom{m-i}{j}y^{(j,-\alpha)}(1-x-y)^{(m-i-j,-\alpha)}$$

$$=(m-i)^{(s)}\sum_{j=s}^{m-i}\binom{m-i-s}{j-s}y^{(j,-\alpha)}(1-x-y)^{(m-i-j,-\alpha)}.$$

If we set $j-s=l$ and take into account formula (1) we obtain

$$S=(m-i)^{(s)}\sum_{l=0}^{m-i-s}\binom{m-i-s}{l}y^{(s+l,-\alpha)}(1-x-y)^{(m-i-s-l,-\alpha)}$$

$$=(m-i)^{(s)}y^{(s,-\alpha)}\sum_{l=0}^{m-i-s}\binom{m-i-s}{l}(y+s\alpha)^{(l,-\alpha)}(1-x-y)^{(m-i-s-l,-\alpha)}.$$

Now if we refer to Vandermonde's formula

$$(a+b)^{(n,h)}=\sum_{k=0}^{n}\binom{n}{k}a^{(k,h)}b^{(n-k,h)}$$

we find that

(12) $$S=(m-i)^{(s)}y^{(s,-\alpha)}(1-x+s\alpha)^{(m-i-s,-\alpha)}.$$

Consequently, by (9)—(12) we can write

$$1^{(m,-\alpha)}M_{(r,s)}=y^{(s)}\sum_{i=r}^{m}i^{(r)}\binom{m}{i}(m-i)^{(s)}x^{(i,-\alpha)}(1-x+s\alpha)^{(m-i-s,-\alpha)}$$

$$=m^{(r)}y^{(s)}\sum_{i=r}^{m}\binom{m-r}{i-r}(m-i)^{(s)}x^{(i,-a)}(1-x+sa)^{(m-i-s,-a)}$$

If we now set $i-r=k$ we obtain

$$1^{(m,-a)}M_{(r,s)}=m^{(r)}y^{(s)}\sum_{k=0}^{m-r}(m-r-k)^{(s)}\binom{m-r}{k}x^{(r+k,-a)}(1-x+sa)^{(m-i-s,-a)}.$$

But by (1) and (10) we have

$$x^{(r+k,-a)}=x^{(r,-a)}(x+ra)^{(k,-a)},$$

$$(m-r-k)^{(s)}\binom{m-r}{k}=(m-r-k)^{(s)}\binom{m-r}{m-r-k}=(m-r)^{(s)}\binom{m-r-s}{s},$$

so that

$$1^{(m,-a)}M_{(r,s)}$$

$$=m^{(r)}(m-r)^{(s)}x^{(r,-a)}y^{(s,-a)}\sum_{k=0}^{m-r-s}\binom{m-r-s}{k}(x+ra)^{(k,-a)}(1-x+sa)^{(m-r-s-k,-a)}.$$

$$=m^{(r)}(m-r)^{(s)}x^{(r,-a)}y^{(s,-a)}(1+\overline{r+s}a)^{(m-r-s,-a)}.$$

Hence we obtain finally

$$M_{(r,s)}=\frac{m^{(r)}(m-r)^{(s)}x^{(r,-a)}y^{(s,-a)}}{1^{(r+s,-a)}}$$

and the insertion of this result into (8) leads us to the required formula (7).

For $a=0$ (7) reduces to a formula established in [11] for the Bernstein polynomial (2).

By making use of the relation between divided differences and finite differences

$$\begin{bmatrix} a, a+h, \dots, a+rh \\ b, b+k, \dots, b+sk \end{bmatrix};f\Bigg]=\frac{1}{r!s!h^rk^s}\Delta^{r,s}_{h,k}f(a,b)$$

we have

$$\Delta^{r,s}_{\frac{1}{m},\frac{1}{m}}f(0,0)=\frac{r!s!}{m^{r+s}}\begin{bmatrix} 0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{r}{m} \\ 0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{s}{m} \end{bmatrix};f\Bigg].$$

Therefore, an immediate consequence of formula (7) is the following

$$(P_m^{[a]}f)(x,y)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}1^{(r+s,m-1)}\frac{x^{(r,-a)}y^{(s,-a)}}{1^{(r+s,-a)}}\begin{bmatrix} 0, \frac{1}{m}, \dots, \frac{r}{m} \\ 0, \frac{1}{m}, \dots, \frac{s}{m} \end{bmatrix};f\Bigg].$$

7. In the next theorem we present the basic result of this paper.

Theorem 2. If $0\leq a=a_m\to 0$ as $m\to\infty$ and f is continuous on the tri)-angle D, then the sequence $(P_m^{[a]}f)$ converges to f uniformly on D.

If $0>\alpha=\alpha_m$ and $-m\alpha_m\leq\varepsilon$, $m=1,2,\ldots$, where $0<\varepsilon<1/3$, then if f is defined and bounded on D and continuous on the triangle D_ε having the vertices (6), then the sequence $(P_m^{[\alpha]})$ converges to f uniformly on D_ε.

P r o o f. It follows easily from (7) that for the functions e_{ij}, $0\leq i+j\leq 2$, defined by $e_j(x,y)=x^iy^j$ for all (x,y) on D, we have

$$(P_m^{[\alpha]}e_{00})(x,y)=1,\ (P_m^{[\alpha]}e_{10})(x,y)=x,\ (P_m^{[\alpha]}e_{01})(x,y)=y,$$

$$(13)\quad (P_m^{[\alpha]}e_{20})(x,y)=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{m}+x(x+\alpha)\right],\quad (P_m^{[\alpha]}e_{11})(x,y)=\left(1-\frac{1}{m}\right)\frac{xy}{1+\alpha},$$

$$(P_m^{[\alpha]}e_{02})(x,y)=\frac{1}{1+\alpha}\left[\frac{y(1-y)}{m}+y(y+\alpha)\right].$$

Therefore, under our assumptions we have

$$\lim_{m\to\infty}(P_m^{[\alpha]}e_{ij})(x,y)=e_{ij}(x,y),\quad 0\leq i+j\leq 2,$$

uniformly on our basic regions.

Since $P_m^{[\alpha]}$ is a linear positive operator on D, respectively on D_ε, the assertions of our theorem now follow by virtue of the Bohman-Korovkin theorem corresponding to two variables (see, e. g., [1], [2] and [3]).

8. We now discuss the estimation of the order of approximation of the function f by means of the operator $P_m^{[\alpha]}$, in order to see the speed of convergence of $(P_m^{[\alpha]}f)$ to f. We shall make use of the modulus of continuity ω on D_ε, defined by

$$\omega(f;\delta)=\omega(\delta)=\sup|f(x'',y'')-f(x',y')|,$$

where $\delta>0$, while (x',y') and (x'',y'') are points from D_ε so that

$$|x''-x'|+|y''-y'|\leq\delta.$$

Consider the space $C(D_\varepsilon)$ of functions continuous on D_ε, and the uniform norm, defined by

$$\|g\|=\|g\|_{D_\varepsilon}=\max\{|g(x,y)|:(x,y)\in D_\varepsilon\}.$$

We shall now establish

Theorem 3. If $f\in C(D_\varepsilon)$ then we have

$$(14)\qquad \|f-P_m^{[\alpha]}f\|\leq 2\omega\left(\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right).$$

P r o o f. It should be clear that if $\alpha\geq 0$ then we take $\varepsilon=0$, and if $\alpha<0$ then we assume that α depends on m in such a way that $-m\alpha\leq\varepsilon$, where $0<\varepsilon<1/3$. In both cases the linear operator $P_m^{[\alpha]}$ is positive on D_ε, since $w_{m,i,j}^{[\alpha]}(x,y)\geq 0$ whenever $(x,y)\in D_\varepsilon$. Taking this into account and the first relation from (13) we can write

$$|f(x,y)-(P_m^{[\alpha]}f)(x,y)|\leq\sum_{0\leq i+j\leq m}w_{m,i,j}^{[\alpha]}(x,y)\left|f(x,y)-f\left(\frac{i}{m},\frac{j}{m}\right)\right|.$$

We shall use the following two properties of the modulus of continuity:

$$|f(x'',y'')-f(x',y')|\leq\omega(|x''-x'|+|y''-y'|),\quad \omega(\lambda\delta)\leq(1+\lambda)\varphi(\delta),\ \lambda>0.$$

Since

$$\left|f(x,y)-f\left(\frac{i}{m},\frac{j}{m}\right)\right|\leq\left(1+\frac{1}{\delta}\left(\left|x-\frac{i}{m}\right|+\left|y-\frac{j}{m}\right|\right)\omega(\delta),\right.$$

it follows that

$$(15)\qquad |f(x,y)-(P_m^{[\alpha]}f)(x,y)|\leq[1+\delta^{-1}(P_m^{[\alpha]}|x-t|)(x,y)$$
$$+\delta^{-1}(P_m^{[\alpha]}|y-\tau|)(x,y)]\omega(\delta),$$

where in the second member the operator $P_m^{[\alpha]}$ is to be performed first with respect to t and then with respect to τ, x and y being fixed.

In accordance with the Cauchy-Schwarz inequality and with the identities (13) we have

$$(16)\qquad (P_m^{[\alpha]}|x-t|)(x,y)\leq[(P_m^{[\alpha]}(x-t)^2(x,y)]^{1/2}$$

$$=[x^2-2x(P_m^{[\alpha]}t)(x,y)+(P_m^{[\alpha]}t^2)(x,y)]^{1/2}=\left[\frac{1+\alpha m}{1+\alpha}\,\frac{x(1-x)}{m}\right]^{1/2}\leq\frac{1}{2}\sqrt{\frac{1+\alpha m}{m+\alpha m}}.$$

Similarly it may be shown that

$$(16')\qquad (P_m^{[\alpha]}|y-t|)(x,y)\leq\frac{1}{2}\sqrt{\frac{1+\alpha m}{m+\alpha m}}.$$

By using these inequalities, (15) leads us to

$$\|f-P_m^{[\alpha]}f\|\leq\left(1+\frac{1}{\delta}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right)\omega(\delta).$$

Now if we set

$$\delta=\delta_m=\sqrt{\frac{1+\alpha m}{m+\alpha m}}$$

we obtain the desired inequality (14).

Specializing to the case $\alpha=0$ one obtains the corresponding inequality for the Bernstein polynomials (2), established first in [5].

9. Our next result yields a way of describing an asymptotic estimate of remainder, of Voronovskaja type, of the approximation formula of the function f by $P_m^{[\alpha]}f$.

Theorem 4. Let f be defined and bounded on D. If $0\leq\alpha=\alpha_m\longrightarrow 0$ and f has on $(x,y)\in D$ the differential $d^2f(x,y)$, then

$$(17)\qquad (P_m^{[\alpha]}f)(x,y)=f(x,y)+\frac{1+\alpha m}{1+\alpha}\left[\frac{x(1-x)}{2m}f''_{xx}(x,y)-\frac{xy}{m}f''_{xy}(x,y)\right.$$
$$\left.+\frac{y(1-y)}{2m}f''_{yy}(x,y)\right]+\frac{\eta_m^{[\alpha]}}{m},$$

where $\eta_m^{[\alpha]}\longrightarrow 0$ as $m\longrightarrow\infty$.

If $\alpha=\alpha_m<0$ and $-m\alpha_m\leq\varepsilon$, $m=1,2,\ldots$, where $0<\varepsilon<1/3$, then if f has on $(x,y)\in D_\varepsilon$ the differential $d^2f(x,y)$, then formula (17) is also valid.

Proof. By virtue of Taylor's theorem we may write the expansion

$$f(t,\tau)=f(x,y)+(t-x)f'_x(x,y)+(\tau-y)f'_y(x,y)$$

$$+\frac{1}{2}(t-x)^2f''_{xx}(x,y)+(t-x)(\tau-y)f''_{xy}(x,y)$$

$$+\frac{1}{2}(\tau-y)^2f''_{yy}(x,y)+(t-x)^2g_1(t,y)+(t-x)(\tau-y)g_2(t,\tau)+(\tau-y)^2g_3(x,\tau),$$

where $(t,\tau)\in D$ and $g_1,g_2,g_3\longrightarrow 0$ at $t\longrightarrow x$ and $\tau\longrightarrow y$.

We see at once that if we set $t=i/m$, $\tau=j/m$, multiply by (3″), sum for i and j such that $0\leqq i+j\leqq m$ and use (13) we obtain

$$(P_m^{[\alpha]}f)(x,y)=f(x,y)+\frac{1+\alpha m}{1+\alpha}\left[\frac{x(1-x)}{2m}f''_{xx}(x,y)-\frac{xy}{m}f''_{xy}(x,y)\right.$$

$$\left.+\frac{y(1-y)}{2m}f''_{yy}(x,y)\right]+\varrho_m^{[\alpha]}(x,y),$$

where

$$\varrho_m^{[\alpha]}(x,y)=\sum_{0\leqq i+j\leqq m}w_{m,i,j}^{[\alpha]}(x,y)\left[\left(\frac{i}{m}-x\right)^2g_1\left(\frac{i}{m},y\right)\right.$$

$$\left.+\left(\frac{i}{m}-x\right)\left(\frac{j}{m}-y\right)g_2\left(\frac{i}{m},\frac{j}{m}\right)+\left(\frac{j}{m}-y\right)^2g_3\left(x,\frac{j}{m}\right)\right].$$

Now, since $g_1,g_2,g_3\longrightarrow 0$ when $t\longrightarrow x$ and $\tau\longrightarrow y$, it follows that given $\eta>0$, we may choose $\delta>0$ so that $|g_i|<\varepsilon$, $i=1,2,3$, whenever t and τ obey the conditions: $|t-x|\leqq\delta$, $|\tau-y|\leqq\delta$. By setting $t=i/m$ and $\tau=j/m$ we should have

(18) $$\left|g_1\left(\frac{i}{m},y\right)\right|<\varepsilon,\quad \left|g_2\left(\frac{i}{m},\frac{j}{m}\right)\right|<\varepsilon,\quad \left|g_3\left(x,\frac{j}{m}\right)\right|<\varepsilon,$$

$$\text{whenever }\left|\frac{i}{m}-x\right|\leqq\delta\ \text{ and }\ \left|\frac{j}{m}-y\right|\leqq\delta.$$

Clearly

(19) $$|\varrho_m^{[\alpha]}(x,y)|\leqq\sum_{0\leqq i+j\leqq m}G_{m,i,j}^{[\alpha]}(x,y),$$

where, for brevity, we used the notation

$$G_{m,i,j}^{[\alpha]}(x,y)=w_{m,i,j}^{[\alpha]}(x,y)\left[\left(\frac{i}{m}-x\right)^2\left|g_1\left(\frac{i}{m},y\right)\right|\right.$$

$$\left.+\left|\left(\frac{i}{m}-x\right)\left(\frac{j}{m}-y\right)\right|\left|g_2\left(\frac{i}{m},\frac{j}{m}\right)\right|+\left(\frac{j}{m}-y\right)^2\left|g_3\left(x,\frac{j}{m}\right)\right|\right].$$

If we divide the set of indices (i, j) into four classes:

$$I_1:\left\{(i,j):0\leqq i+j\leqq m,\left|\frac{i}{m}-x\right|\leqq\delta,\left|\frac{j}{m}-y\right|\leqq\delta\right\},$$

$$I_2:\left\{(i,j):0\leqq i+j\leqq m,\left|\frac{i}{m}-x\right|\leqq\delta,\left|\frac{j}{m}-y\right|\geqq\delta\right\},$$

$$I_3 : \left\{(i,j) : 0 \leq i+j \leq m, \left|\frac{i}{m} - x\right| \geq \delta, \left|\frac{j}{m} - y\right| \leq \delta\right\},$$

$$I_4 : \left\{(i,j) : 0 \leq i+j \leq m, \left|\frac{i}{m} - x\right| \geq \delta, \left|\frac{j}{m} - y\right| \geq \delta\right\}$$

and split the sum from (19) in a corresponding way we have:

$$|\varrho_m^{[\alpha]}(x,y)| \leq \sum_{\nu=1}^{4} \sum_{(i,j)\in I_\nu} G_{m,i,j}^{[\alpha]}(x,y).$$

Further if we take into consideration the foregoing results and proceed as in [5] one sees that $\varrho_m^{[\alpha]}(x,y) = \frac{1}{m}\eta_m^{[\alpha]}(x,y)$, where $\eta_m^{[\alpha]}(x,y) \longrightarrow 0$ as $m \longrightarrow \infty$ and that this convergence is uniform on D (resp. on D_ε) if the function f has all its second-order partial derivatives continuous on D (resp. on D_ε), under the assumptions made on the parameter α.

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