EVALUATION OF THE REMAINDERS IN CERTAIN APPROXIMATION PROCEDURES BY MEYER-KÖNIG AND ZELLER-TYPE OPERATORS

by D. D. Stancu in Cluj

1. In our previous paper [9] we have introduced and studied a parameter-dependent linear positive operator $W_m^{\langle\alpha\rangle}$ $(m = 1, 2, \ldots)$, of Meyer-König and Zeller-type. Denoting by $u^{(n,h)}$ the factorial power of order n $(n \geq 0)$ and increment h of u, that is

$$u^{(n,h)} = u(u-h)\ldots(u-\overline{n-1}\,h), \quad u^{(0,h)} = 1,$$

the operator $W_m^{\langle\alpha\rangle}$, associated to a function $f : [0,1] \to \mathbb{R}$, can be defined by

$$(W_m^{\langle\alpha\rangle}f)(x) = \sum_{k=0}^{\infty} w_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m+k}\right),$$

where $x \in [0,1)$ and

$$w_{m,k}^{\langle\alpha\rangle}(x) = \binom{m+k}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+k+1,-\alpha)}},$$

α being a non-negative parameter.

If $x = 0$ it is easy to see that we have $(W_m^{\langle\alpha\rangle}f)(0) = f(0)$, while if $x = 1$ it is convenient to define

$$(W_m^{\langle\alpha\rangle}f)(1) = \lim_{x \uparrow 1} (W_m^{\langle\alpha\rangle}f)(x) = f(1).$$

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The operator $W_m^{\langle\alpha\rangle}$ includes as a special case $(\alpha = 0)$ the well known operator M_m of MEYER-KÖNIG and ZELLER [3], defined by

$$(M_m f)(x) = \sum_{k=0}^{\infty} w_{m,k}(x) f\left(\frac{k}{m+k}\right), \quad w_{m,k}(x) = \binom{m+k}{k} x^k (1-x)^{m+1},$$

obtained by these authors using the negative binomial (Pascal) probability distribution.

In a recent paper [12], J. SWETITS and B. WOOD, refering to a probabilistic method in connection with the Markov-Pólya urn scheme used by us [8] for constructing a class of a parameter-dependent linear polynomial operators [7] have presented a variation of the Pascal urn scheme, in the sense in which Markov and Pólya have generalized the Bernoulli urn scheme. It enable them to obtain by a probabilistic way the operator $W_m^{\langle\alpha\rangle}$.

As we have shown in [9], if $\alpha > 0$ and $0 < x < 1$, then we can represent the operator $W_m^{\langle\alpha\rangle}$ by means of the operator M_m in the following form

$$(W_m^{\langle\alpha\rangle} f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}} (1-t)^{\frac{1-x}{\alpha}-1} (M_m f)(t)\,dt,$$

where B is the beta function.

2. The purpose of this paper is to evaluate the remainder term of the approximation formula

$$(1) \qquad f(x) = (W_m^{\langle\alpha\rangle} f)(x) + (R_m^{\langle\alpha\rangle} f)(x).$$

First we give a representation of this remainder in terms of divided differences of second-order of f.

THEOREM 1. *The remainder of the approximation formula* (1) *can be represented in the following form*

$$(2) \qquad (R_m^{\langle\alpha\rangle} f)(x) = -\sum_{k=0}^{\infty} \frac{(x+k\alpha)(1-x+m\alpha)}{(m+k+1)(1+m+k\alpha)}\, w_{m-1,k}^{\langle\alpha\rangle}(x) \left[x, \frac{k}{m+k}, \frac{k+1}{m+k+1}; f\right],$$

where the brackets represent the symbol for divided differences.