for $i = i_o$, (for $j = j_o$); then consider $I_h^{t+1} = I_h^t \cup \{i_o\}$, ($J_h^{t+1} = J_h^t \cup \{j_o\}$), $J_h^{t+1} = J_h^t$, ($I_h^{t+1} = I_h^t$), $S_h^{t+1} = \{(i,j) \mid i \in I_h^{t+1}, j \in J_h^{t+1}\}$, and go to Step 3.

<u>S t e p 5</u>. Suppose we have obtained the sets $S_h^{t_h}$, $h = 1,\dots,r$, ($t_h \geq 0$), and consider $\mathcal{M} = \{S_1^{t_1},\dots,S_r^{t_r}\}$; if there exist i,j, ($i \neq j$), such that $S_i^{t_i} \subset S_j^{t_j}$, then delate $S_j^{t_j}$ from $\mathcal{M}$. After these delations we obtain a new set $\overline{\mathcal{M}} = \{S_{i_1}^{t_{i_1}},\dots,S_{i_k}^{t_{i_k}}\}$, the spaning subgraphs $[S_{i_1}^{t_{i_1}}],\dots,[S_{i_k}^{t_{i_k}}]$ being all generalized saddle points of G.

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<u>AN EXTREMAL PROBLEM IN THE THEORY OF NUMERICAL QUADRATURES WITH MULTIPLE NODES</u>

by D.D.Stancu

In this paper we shall develop some theory and algoritms for constructing a general quadrature formula using multiple nodes prescribed in advance and other multiple nodes which we select in order to increase the corresponding degree of exactness.

Assume that we have a definite integral of the form

$$I(w;f) = I(f) = \int_a^b w(x)f(x)dx, \tag{1}$$

where (a,b) is a bounded or unbounded interval of the real axis, w is a given weight function for which the moments $c_k = I(e_k)$ $[e_k(x) = x^k \ \forall x \in (a,b)]$ $(k=0,1,2,\dots)$ exist and $c_0 > 0$, while f is a real-valued function having continuous derivatives of whatever orders will be needed.

We consider the fixed nodes a_i, given with their orders of multiplicities $m_i + 1$ $(i = 0, 1, \dots, r)$, so that the polynomial ω, defined by

$$\omega(x) = \alpha \prod_{i=0}^{r} (x-a_i)^{m_i+1}, \tag{2}$$

is non-negative on (a,b) for some constant $\alpha \neq 0$.

We denote by x_k some nodes multiple of given odd orders of multiplicities n_k+1 $(k=0,1,\dots,s)$.

We next investigate a general quadrature formula of the form

$$I(f) = \phi(f) + R(f), \tag{3}$$

where

$$\phi(f) = \sum_{i=0}^{r}\sum_{j=0}^{m_i} A_{i,j} f^{(j)}(a_i) + \sum_{k=0}^{s}\sum_{l=0}^{n_k} B_{k,l} f^{(l)}(x_k), \tag{4}$$

the coefficients $A_{i,j}$ and $B_{k,l}$ being independent on f, while the remainder $R(f)$ is, by definition, the difference $I(f) - \phi(f)$.

Let us introduce the notations

$$M = \sum_{i=0}^{r}(m_i+1), \quad N = \sum_{k=0}^{s}(n_k+1). \tag{5}$$

One says that formula (3) has the degree of exactness D if we have $R(e_j)=0$ $(j=0,1,\dots,D)$ and $R(e_{D+1}) \neq 0$. We seek to determine the nodes x_k so that for appropriate values of the coefficients $A_{i,j}$ and $B_{k,l}$ the degree D is as large as possible.

The following theorem is the central result of this paper.

Theorem. A necessary and sufficient condition that the quadrature formula (3) has the highest degree of exactness, namely $D = M+N+s$, is that the polynomial u defined by

$$(6) \qquad u(x) = \prod_{k=0}^{s} (x-x_k)^{n_k+1} ,$$

be orthogonal, with respect to the weight function $w\omega$, to any polynomial of degree $\leq s$. There exists at least one quadrature formula of the form (3) having the highest degree of exactness $D=M+N+s$.

Proof. Let us introduce the notation

$$(7) \qquad U(x) = \prod_{k=0}^{s} (x-x_k)^{n_k+2} .$$

We have $\Phi(\omega U) = 0$ and $I(\omega U) \neq 0$ since $\omega(x)U(x) \geq 0$ $\forall x \in (a,b)$ and it is not identically zero. It follows that the highest degree of exactness D of (3) must satisfy the inequality $D \leq M+N+s$. We shall prove that one can choose the nodes x_k so that $D = M+N+s$.

In order to construct a quadrature formula of the form (3) we first introduce $s+1$ distinct real nodes γ_j $(j=0,1,\ldots,s)$, assuming that none of these coincide with any of the a_i and x_k. Then we use the Lagrange-Hermite interpolation formula corresponding to the function f and the points a_i, x_k and γ_j:

$$(8) \qquad f(x) = (L_H f)(x) + (rf)(x) ,$$

where the interpolating polynomial is

$$(L_H f)(x) = (L_H f)\left(x; \begin{matrix} a_i \\ m_i+1 \end{matrix}, \begin{matrix} x_k \\ n_k+1 \end{matrix}, \begin{matrix} j \\ 1 \end{matrix}\right),$$

the numbers beneath the nodes designating their multiplicities.

It is easily verified that we have

$$(9) \qquad (L_H f)(x) = v(x)(L_H f_1)\left(x; \begin{matrix} a_i \\ m_i+1 \end{matrix}, \begin{matrix} x_k \\ n_k+1 \end{matrix}\right) + \omega(x)u(x)(L_H f_2)\left(x; \begin{matrix} \gamma_j \\ 1 \end{matrix}\right),$$

where

$$(10) \qquad v(x) = \prod_{j=0}^{s}(x-\gamma_j) , \quad f_1(x) = \frac{f(x)}{v(x)} , \quad f_2(x) = \frac{f(x)}{\omega(x)u(x)} .$$

The remainder $r(rf)(x)$ of interpolation formula (8) can be expressed by a divided difference as follows

$$(11) \qquad (rf)(x) = \alpha^{-1}\omega(x)u(x)v(x)\left[\begin{matrix} a_i \\ m_i+1 \end{matrix}, \begin{matrix} x_k \\ n_k+1 \end{matrix}, \begin{matrix} \gamma_j \\ 1 \end{matrix}, \begin{matrix} x \\ 1 \end{matrix}; f\right].$$

By integrating the preceding interpolation formula we obtain a quadrature formula of the following form

$$(12) \qquad I(f) = \Phi(f) + \Omega(f) + \rho(f) ,$$

where

$$(13) \qquad \rho(f) = I\big((rf)(x)\big)$$

and $\Omega(f)$ has the form

$$\Omega(f) = \sum_{j=0}^{s} C_j f(\gamma_j) .$$

Of course, the coefficients $A_{i,j}$ and $B_{k,\ell}$, which appear in the preceding quadrature formula, are independent of the function f.

Because the divided difference which occurs in (11) is of order $M+N+s+1$, it follows that the quadrature formula (12) has the degree of exactness $D=M+N+s$.

Now we want to try to determine the nodes x_k so that $C_j = 0$ $(j=0,1,\ldots,s)$. Because

$$C_j = \int_a^b \frac{w(x)\omega(x)u(x)v_j(x)}{\omega(\gamma_j)u(\gamma_j)v_j(\gamma_j)}dx ,$$

where

$$v_j(x) = \frac{v(x)}{x-\gamma_j} ,$$

we observe that we should have $I(w;\omega u v_j)=0$ $(j=0,1,\ldots,s)$.

Taking into account the fact that γ_j are arbitrary, it follows that $C_j=0$ $(j=0,1,\ldots,s)$ if and only if the polynomial $u(x)$ is orthogonal on (a,b), with respect to the weight function $w\omega$, to all polynomials of degree $\leq s$. This is equivalent to

$$(14) \qquad I(w\omega; e_j u) = 0 \quad (j=0,1,\ldots,s) .$$

Here we have a system of $s+1$ equations in $s+1$ unknowns: $x_0,\ldots,x_s$.

One can prove that this system has at least a real solution, by considering the function

$$F(t_0,t_1,\dots,t_s) = I(w;U) = \tag{15}$$

$$= \int_a^b w(x)\omega(x)(x-t_0)^{n_0+2}\dots(x-t_s)^{n_s+2}\,dx \ ,$$

which is defined in the Euclidean space R_{s+1}.

This function, which is continuous and positive, has a lower bound μ_0 on the domain: $-\infty<t_0\leq t_1\leq\dots\leq t_s<\infty$ of the space R_{s+1}.

Using the results from [1], [2], [5], one can prove that the value μ_0 is taken by the function F at least in a point of distinct coordinates $(x_0,x_1,\dots,x_s)$, these coordinates being situated in the interior of the interval.

Since at this point there is a relative minimum, we have

$$-\frac{1}{n_k+2}\cdot\frac{\partial F}{\partial x_k} = I(P_k)=0, \tag{16}$$

where

$$P_k(x) = \omega(x)\frac{U(x)}{x-x_k} \ .$$

It follows that we must have

$$I(\omega u l_k) = 0 \quad (k=0,1,\dots,s) \ , \tag{17}$$

where l_k are the fundamental interpolation polynomials of Lagrange with regard to the distinct points $x_0,x_1,\dots,x_s$.

Because l_k are linear independent, it follows that the polynomial u satisfies the orthogonality condition (14).

If we now assume that the system of real numbers $(x_0,\dots,x_s)$ satisfies the conditions (14), then it follows immediately that their orders of multiplicities $r_0+1,\dots,r_s+1$ are odd numbers.

If we take into account that

$$\frac{\partial F}{\partial x_k}=0 \ , \quad \frac{\partial^2 F}{\partial x_k\partial x_l}=0 \ , \quad \frac{\partial^2 F}{\partial x_k^2}>0 \ ,$$

$$(k,l=0,1,\dots,s;\ k\neq l)$$

we conclude that at the point $(x_0,\dots,x_s)$, for which the conditions (14) are fulfilled, the function F has a relative minimum.

As regards to the remainder, if we assume that the function f

has a continuous derivative of order $M+N+s+1$ on (a,b), and we make $\gamma_j\to x_j$ $(j=0,1,\dots,s)$, then formula (11) permits us to give the following expression for the remainder of the general quadrature formula (3)-(4):

$$R(f) = -\frac{f^{(M+N+s+1)}(\xi)}{(M+N+s+1)!}\frac{1}{\alpha}I(w;\omega U) \ , \quad \xi\in(a,b) \ .$$

We should mention that related papers to our work are: [6], [1], [5], [2], [3], [4] .

We shall now give two examples.

1^o. If $(a,b)=(-1,1)$, $p(x)=(1-x)^\alpha(1+x)^\beta$ $(\alpha,\beta>-1)$, $a_0=-1$, $a_1=1$ are simple fixed nodes and x_0 is a simple arbitrary node, then the highest degree of exactness $D=(1+1)+1+0=3$ is achieved if

$$x_0 = \frac{\beta-\alpha}{\alpha+\beta+4}$$

and the corresponding quadrature formula is

$$\int_{-1}^1 (1-x)^\alpha(1+x)^\beta f(x)\,dx =$$

$$= 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+2)(\beta+2)\Gamma(\alpha+\beta+4)}\Big[(\alpha+1)(\alpha+2)^2 f(-1)+$$

$$+(\alpha+1)(\beta+1)(\alpha+\beta+4)^2 f\Big(\frac{\beta-\alpha}{\alpha+\beta+4}\Big)+(\beta+1)(\beta+2)^2 f(1)\Big]-$$

$$-2^{\alpha+\beta+2}\frac{\Gamma(\alpha+3)\Gamma(\beta+3)}{3(\alpha+\beta+4)\Gamma(\alpha+\beta+6)}f^{IV}(\xi),$$

which for $\alpha=\beta=0$ becomes the classical Cavalieri-Simpson quadrature formula.

2^o. If $(a,b)=(-1,1)$, $p(x)=1$, $\forall x\in(-1,1)$ and we have no fixed nodes, but we have two variable nodes: x_0 (simple) and x_1 (triple), then the highest degree of exactness $D=0+1+3+1=5$ is attained if we have $x_0=-\sqrt{5}/3$, $x_1=\sqrt{5}/5$, the corresponding quadrature formula being

$$\int_{-1}^1 f(x)\,dx = \frac{1}{128}\Big[81 f\Big(-\frac{\sqrt5}{3}\Big)+175 f\Big(\frac{\sqrt5}{5}\Big)-\frac{40}{\sqrt5}f'\Big(\frac{\sqrt5}{5}\Big)+\frac{32}{3}f''\Big(\frac{\sqrt5}{5}\Big)\Big]$$

$$+\frac{8}{70875}f^{VI}(\xi),$$

constructed earlier in [1], [2], and [5], by using other methods.

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ETUDE A L'EGARD DE L'AMELIORATION DU SERVICE AVEC L'ENERGIE ELECTRIQUE POUR LES CONSOMMATEURS DOMESTIQUES

Adrian A.Stănculescu

Dans le rapport qui suit on presente une méthode par laquelle nous pouvons obtenir une amélioration du service avec l'énergie électrique pour les consommateurs domestiques, en suivant la reduction du temps de service pour le consommateur (qui se nomme içi "le client") avec une perte minime en regardent le consommation de combustible (pour les voitures d'intervention) qui appartient de l'usine (ç'est à dire pour "le fournisseur").

L'analyse a été effectué dans une usine de reseaux électriques, pendant une periode du temps.

Les clients suffisent parfois des interruptions de l'énergie électrique, la raison étant les derangements qui apparaissent dans les reseaux.

Par consequent, le dérangement est annoncé à l'un de l'apparails télephonique qui existe monté au siège du bureau de distribution avec l'énergie électrique à l'usine.

Içi se trouve en permanence un fonctionnaire de service qui peut receptionner ce dérangement.

Le dérangement qui a été annoncé est enregistré dans le registre de dérangement, puis le fonctionnaire en transmet directemment ou par le radio chez une voiture de service qui est libre (Le service de l'usine est doté avec deux voitures de service pour remedier les dérangemments).

Apres quoi le dérangemment a été remedié en terrain, on transmet chez l'usine le fin du travail et l'alimentation du consommateur avec l'énergie électrique.

Tous les temps de ces operations on ecrit dans le registre.