QUADRATURE FORMULAS CONSTRUCTED BY USING CERTAIN LINEAR POSITIVE OPERATORS

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1. Let [a,b] be a compact interval of the real line $\mathbb{R}$. It is known that the classical theorem of Bohman-Korovkin states that in order that a sequence of positive linear operators (L_m), mapping into itself the space $C[a,b]$ of continuous real-valued functions on [a,b], equipped with the uniform norm, to have the property that, for any $f \in C[a,b]$, if $m \to \infty$ we have $\lim L_m f = f$, uniformly on [a,b], it is necessary and sufficient that such a convergence occur for a triplet of "test functions" from C[a,b], forming a so called Korovkin system. For C[a,b] the three monomials e_0, e_1, e_2, where $e_j(x) := x^j$ $(j = 0,1,2)$, represent such a system.

In the sequel we shall consider linear positive operators defined, for any $f \in C[a,b]$, by a formula of the following form

(1) $$(L_m f)(x) := \sum_{k=0}^{m} q_{m,k}(x) f(x_{m,k}) ,$$

where the nodes $x_{m,k}$ are distinct points from [a,b], while $q_{m,k}$ are non-negative polynomials on [a,b], for any $k=0(1)m$ and $m \in \mathbb{N}$. Such operators are usually called interpolation operators, in the sense that the values of the function f at a finite number of points can be used to calculate the values of $L_m f$.

The best known example of linear positive interpolation operator is represented by the classical Bernstein operator, defined by

(2) $$(B_m f)(x) := \sum_{k=0}^{m} p_{m,k}(x) f(x_{m,k}) ,$$

G. Hämmerlin (ed.), *Numerical Integration*

where

(3) $$p_{m,k}(x) := \binom{m}{k} x^k (1-x)^{m-k}, \quad x_{m,k} := \frac{k}{m} .$$

By using the linear interpolation of Lagrange, we have constructed in [7] a Bernstein type polynomial of the form (2), where

(4) $$x_{m,k} := \frac{k+\alpha}{m+\beta}, \quad k = 0(1)m ,$$

α and β being real parameters which satisfy the conditions: $0 \leq \alpha \leq \beta$.

Generalizations of the operator B_m have been given, for instance, in the papers: [2], [5], [1] and [3].

A very interesting positive linear interpolation operator is the operator of Hermite-Fejér F_{2m+1}, defined, for any $f \in C[-1,1]$, by

(5) $$(F_{2m+1}f)(x) := \sum_{k=0}^{m} h_{m,k}(x) f(x_k) ,$$

where

(6) $$h_{m,k}(x) := (1-x_k x)\left(\frac{T_{m+1}(x)}{(m+1)(x-x_k)}\right)^2 ,$$

$$T_{m+1}(x) := \cos[(m+1) \arccos x] ,$$

while the nodes x_k are the roots of the Chebyshev orthogonal polynomial of the first kind T_{m+1} .

The operator B_m reproduces the linear functions, but the operator F_{2m+1} reproduces only the constants. It is known that in general a positive linear operator L_m of Bernstein type cannot preserve the quadratic polynomials only if it coincides with the identity operator I .

2. By using the operator L_m, defined at (1), we can construct a quadrature formula, for a weighted integral, of the following form:

(7) $$J(w;f) = \int_a^b w(x) f(x) dx = \sum_{k=0}^{m} A_{m,k} f(x_{m,k}) + R_m(w;f) ,$$