APPROXIMATION OF FUNCTIONS BY MEANS OF A NEW GENERALIZED BERNSTEIN OPERATOR

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Dedicated to Professor Aldo Ghizzetti on his 75th birthday

ABSTRACT - In this paper we first use a probabilistic method to construct a linear positive polynomial operator $L_{m,r}^{\alpha,\beta}$ of Bernstein type, depending on a non-negative integer parameter r and on two real parameters α and β, such that $0 \leq \alpha \leq \beta$. Then we investigate the approximation properties of this operator mapping into itself the Banach space $C[0,1]$ of real-valued continuous functions on $[0,1]$. A special attention is accorded to the case of the operator $L_{m,r} = L_{m,r}^{0,0}$. We prove that the remainder of the approximation formula of a function $f \in C[0,1]$ by $L_{m,r}f$ can be represented either by means of divided differences, or in an integral form, obtained by using a classical theorem of Peano. We give also an asymptotic estimate for this remainder. The operator $L_{m,r}$ enjoys the variation diminishing property — in the sense of I. J. Schoenberg [15]. By extending the known inequalities of T. Popoviciu [12] and G. G. Lorentz [7], we evaluate the orders of approximation in terms of the modulus of continuity of the function f or of its derivative. In the last section of this paper we determine the point spectrum of the operator $L_{m,r}$ and, finally, we present a quadrature formula which can be constructed by means of this operator.

1. Construction of the operators $L_{m,r}^{\alpha,\beta}$ by a probabilistic method.

Consider a succession of independent trials with two possible outcomes: a «success» — with probability p, or a «failure» — with probability $q = 1 - p$.

Let r be a given non-negative integer, m a natural number such that $m > 2r$ and k a non-negative integer satisfying the condition $k \leq m$.

One denotes by $E_{m,k,r}$ the event that at the trial number $m-r+1$ occur the $(m-r-k+1)$ *th* failure or the $(k-r+1)$ *th* success.

It is easy to see that this event occurs if and only if:

(i) among the first $m-r$ trials there are exactly k successes and $m-r-k$ failures and the $(m-r+1)$ *th* trial results in a failure; one denotes this event by $E'_{m,k,r}$, or

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(ii) among the first $m-r$ trials there are exactly $k-r$ successes and $m-k$ failures and the $(m-r+1)$ *th* trial results in a success; one denotes this event by $E''_{m,k,r}$.

We observe that we have

$$E_{m,k,r}=\begin{cases} E'_{m,k,r} & \text{if } 0\leq k<r \\ E'_{m,k,r}\cup E''_{m,k,r} & \text{if } r\leq k\leq m-r \\ E''_{m,k,r} & \text{if } m-r<k\leq m. \end{cases}$$

Consequently we can write for the probability of this event the following formulae

$$P\,(E_{m,k,r})=\pi_{m,k,r}= \tag{1.1}$$

$$=\begin{cases} \binom{m-r}{k}p^k\,q^{m-r-k+1} & \text{if } 0\leq k<r \\ \binom{m-r}{k}\,p^k\,q^{m-r-k+1}+\binom{m-r}{k-r}p^{k-r+1}\,q^{m-k} & \text{if } r\leq k\leq m-r \\ \binom{m-r}{k-r}p^{k-r+1}\,q^{m-k} & \text{if } m-r<k\leq m. \end{cases}$$

Let X be a random variable, of discrete type, which takes on the values $k\,(0\leq k\leq m)$ with the probability $\pi_{m,k,r}$, that is

$$P\,(X=k)=\pi_{m,k,r}\quad (0\leq k\leq m). \tag{1.2}$$

We have

$$\sum_{k=0}^{m}\pi_{m,k,r}=q\sum_{k=0}^{m-r}\binom{m-r}{k}\,p^k\,q^{m-r-k}+p\sum_{k=r}^{m}\binom{m-r}{k-r}p^{k-r}\,q^{m-k}=1.$$

One observes that for $r=0$ or $r=1$ the above probability distribution reduces to the binomial distribution.

Let us consider now a function $f\in C\,[0,1]$ and an associate random variable $Y^f_{\alpha,\beta}$ with the probability distribution given by

$$P\left(Y^f_{\alpha,\beta}=f\left(\frac{k+\alpha}{m+\beta}\right)\right)=\pi_{m,k,r}\quad (0\leq k\leq m),$$

where α and β are given real constants such that $0\leq\alpha\leq\beta$.

The expected value of this random variable is

$$E\,(Y^f_{\alpha,\beta}) = \sum_{k=0}^{m} \pi_{m,k,r}\, f\left(\frac{k+\alpha}{m+\beta}\right).$$

In this way we arrive naturally at a linear positive operator $L^{\alpha,\beta}_{m,r}$ defined by

$$(L^{\alpha,\beta}_{m,r} f)\,(x) := \sum_{k=0}^{m} w_{m,k,r}\,(x)\, f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.3}$$

where $x \in [0, 1]$ and

$$w_{m,k,r}\,(x) = \begin{cases} \binom{m-r}{k} x^k\,(1-x)^{m-r-k+1} & \text{if} \quad 0 \le k < r \\ \binom{m-r}{k} x^k\,(1-x)^{m-r-k+1} + & \text{if} \quad r \le k \le m-r \\ \binom{m-r}{k-r} x^{k-r+1}\,(1-x)^{m-k} & \\ \binom{m-r}{k-r} x^{k-r+1}\,(1-x)^{m-k} & \text{fi} \quad m-r < k \le m. \end{cases} \tag{1.4}$$

This operator represents a generalization, depending on the non-negative integer parameter r and on two real parameters α and β $(0 \le \alpha \le \beta)$, of the classical operator B_m of Bernstein, defined by

$$(B_m f)\,(x) := \sum_{k=0}^{m} p_{m,k}\,(x)\, f\left(\frac{k}{m}\right),$$

where

$$p_{m,k}\,(x) := \binom{m}{k} x^k\,(1-x)^{m-k}. \tag{1.5}$$

It is obvious that $L^{0,0}_{m,0} = L^{0,0}_{m,1} = B_m$. The operator $L^{0,0}_{m,2}$ has been given earlier by H. Brass [2]. In our previous paper [18] we have investigated in detail the operator of Bernstein type $B^{\alpha,\beta}_m = L^{\alpha,\beta}_{m,0} = L^{\alpha,\beta}_{m,1}$.

Because we can write

$$(L^{\alpha,\beta}_{m,r} f)\,(x) = \sum_{k=0}^{m-r} \binom{m-r}{k} x^k\,(1-x)^{m-r+1-k} f\left(\frac{k+\alpha}{m+\beta}\right) +$$

$$+ \sum_{k=r}^{m} \binom{m-r}{k-r} x^{k-r+1}\,(1-x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right),$$

by making use of the following change of index of summation $k-r=j$, we can see immediately that we are able to express our operator by means of the fundamental Bernstein polynomials (1.5) in the following form

$$(1.6)\qquad (L^{\alpha,\beta}_{m,r} f)(x) = \sum_{k=0}^{m-r} P_{m-r,k}(x)\left[(1-x)\, f\left(\frac{k+\alpha}{m+\beta}\right) + x f\left(\frac{k+r+\alpha}{m+\beta}\right)\right].$$

By making use of a method presented in our earlier paper [18] it can be easily shown that we have the following representation by means of finite differences

$$(1.7)\qquad (L^{\alpha,\beta}_{m,r} f)(x) = \sum_{k=0}^{m-r} \binom{m-r}{k}\left[(1-x)\,(\Delta^k_{\frac{1}{m+\beta}} f)\left(\frac{\alpha}{m+\beta}\right) + \right.$$

$$\left. + x\,(\Delta^k_{\frac{1}{m+\beta}} f)\left(\frac{r+\alpha}{m+\beta}\right)\right] x^k.$$

2. The convergence of the sequence $(L^{\alpha,\beta}_{m,r})$.

Since $(L^{\alpha,\beta}_{m,r})$ is a sequence of linear positive operators, by applying the well known theorem of Bohman-Korovkin we can state.

THEOREM 2.1. *If* $f \in C[0,1]$, *r is a non-negative fixed integer and* $0 \le \alpha \le \beta$, *then we have*

$$\lim_{m \to \infty} L^{\alpha,\beta}_{m,r} f = f,$$

uniformly on the interval $[0,1]$.

PROOF. Let us calculate first the value of our operator for the test functions e_0, e_1, e_2, where $e_j(t) := t^j$ $(j=0,1,2)$ $\forall\, t \in [0,1]$.

We have

$$(2.1)\qquad (L^{\alpha,\beta}_{m,r} e_0)(x) = (1-x+x)\sum_{k=0}^{m-r} p_{m-r,k}(x) = 1,$$

$$(2.2)\qquad (L^{\alpha,\beta}_{m,r} e_1)(x) = \sum_{k=1}^{m-r} p_{m-r,k}(x)\frac{k}{m+\beta} + \frac{\alpha+rx}{m+\beta}\sum_{k=0}^{m-r} p_{m-r,k}(x) =$$

$$= \frac{m-r}{m+\beta}\,x + \frac{\alpha+rx}{m+\beta} = x + \frac{\alpha-\beta x}{m+\beta},$$

because

$$\sum_{k=1}^{m-r} p_{m-r,k}(x)\frac{k}{m+\beta} = \frac{m-r}{m+\beta}(B_{m-r}\, e_1)(x) = \frac{m-r}{m+\beta}\cdot x.$$

For the monomial e_2 we can write successively

$$(L_{m,r}^{\alpha,\beta}\, e_2)\,(x)=\sum_{k=0}^{m-r} p_{m-r,k}\,(x)\left[(1-x)\left(\frac{k+\alpha}{m+\beta}\right)^2+x\left(\frac{k+r+\alpha}{m+\beta}\right)^2\right]=$$

$$=\frac{\alpha^2+r\,(r+2\alpha)\,x}{(m+\beta)^2}+2\frac{(m-r)\,(\alpha+rx)}{(m+\beta)^2}\,(B_{m-r}\,e_1)\,(x)+$$

$$+\frac{(m-r)^2}{(m+\beta)^2}(B_{m-r}\,e_2)\,(x)=$$

$$=\frac{1}{(m+\beta)^2}\,[\alpha^2+r\,(r+2\alpha)x+2\,(m-r)\,(\alpha+rx)\,x+(m-r)^2\,x^2+(m-r)\,x\,(1-x)],$$

since

$$(B_{m-r}\,e_2)\,(x)=x^2+\frac{x\,(1-x)}{m-r}.$$

It is helpful to write this result under the form

$$(L_{m,r}^{\alpha,\beta}e_2)\,(x)=x^2+ \tag{2.3}$$

$$+\frac{1}{(m+\beta)^2}\{[m+r\,(r-1)]\,x\,(1-x)+(\alpha-\beta x)\,(2mx+\beta x+\alpha)\}.$$

According to (2.1), (2.2) and (2.3) we have

$$\lim_{m\to\infty} L_{m,r}^{\alpha,\beta}\,e_j=e_j\quad (j=0,1,2),$$

uniformily on $[0,1]$, so that if we invoke the Bohman-Korovkin theorem (see, e. g. [4] or [7]), we obtain the desired result.

3. Evaluation of the remainder.

Let us consider the approximation formula

$$f\,(x)=(L_{m,r}^{\alpha,\beta}\,f)\,(x)+(R_{m,r}^{\alpha,\beta}\,f)\,(x). \tag{3.1}$$

Now referring to (1.3) and (1.4) we can see that

$$(L_{m,r}^{\alpha,\beta}\,f)\,(0)=f\left(\frac{\alpha}{m+\beta}\right),\quad (L_{m,r}^{\alpha,\beta}\,f)\,(1)=f\left(\frac{m+\alpha}{m+\beta}\right),$$

so that if $0=\alpha<\beta$ then our approximation polynomial $L_{m,r}^{\alpha,\beta}\,f$ is interpolatory at the left side of $[0,1]$ and if $0<\alpha=\beta$ then it is interpolatory at the right side only.

If $\alpha=\beta=0$ then we obtain the operator $L_{m,r}=L_{m,r}^{0,0}$, defined by

$$(3.2)\qquad (L_{m,r}f)(x):=\sum_{k=0}^{m} w_{m,k,r}(x)\,f\left(\frac{k}{m}\right)=$$

$$=\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)\,f\left(\frac{k}{m}\right)+xf\left(\frac{k+r}{m}\right)\right],$$

where the fundamental polynomials $w_{m,k,r}$ and $p_{m-r,k}$ are defined at (1.4), respectively at (1.5).

In this case it is easy to see that

$$(3.3)\qquad (L_{m,r}f)(0)=f(0),\ (L_{m,r}f)(1)=f(1).$$

Hence the polynomial $L_{m,r}f$ is interpolatory at both sides of $[0, 1]$.

We notice that if $0<\alpha<\beta$ then our approximation polynomial is not interpolatory at any point of $[0, 1]$ and in this case the degree of exactness of the approximation formula (3.1) equals -1, i. e. the remainder vanishes if and only if f is a polynomial identically zero on $[0, 1]$.

The highest degree of exactness is 1 and it is achieved if and only if $\alpha=\beta=0$. In this case e_0 and e_1 are fixed elements of the operator $L_{m,r}$.

We shall henceforth investigate the remainder term of the approximation formula

$$(3.4)\qquad f(x)=(L_{m,r}f)(x)+(R_{m,r}f)(x)$$

having the maximum degree of exactness.

We now state and prove a basic result of this section.

THEOREM 3.1. *The remainder of the approximation formula* (3.4) *can be expressed by means of the second-order divided differences in the following form*

$$(3.5)\qquad (R_{m,r}f)(x)=$$

$$-\frac{x(1-x)}{m^2}\left\{(m-r)\sum_{k=0}^{m-r-1} p_{m-r-1,k}(x)\left((1-x)\left[x,\frac{k}{m},\frac{k+1}{m};f\right]+\right.\right.$$

$$\left.\left.+x\left[x,\frac{k+r}{m},\frac{k+r+1}{m};f\right]\right)+r^2\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[x,\frac{k}{m},\frac{k+r}{m};f\right]\right\}.$$

PROOF. We can write successively

$$
\begin{aligned}
&(R_{m,r}f)(x)=f(x)-(L_{m,r}f)(x)=\\
&=(1-x)\sum_{k=0}^{m-r}p_{m-r,k}(x)\left(f(x)-f\left(\frac{k}{m}\right)\right)+x\sum_{k=0}^{m-r}p_{m-r,k}(x)\left(f(x)-f\left(\frac{k+r}{m}\right)\right)=\\
&=\frac{1-x}{m}\sum_{k=0}^{m-r}p_{m-r,k}(x)\,(mx-k)\left[x,\frac{k}{m};f\right]+\\
&+\frac{x}{m}\sum_{k=0}^{m-r}p_{m-r,k}(x)\,(mx-r-k)\left[x,\frac{k+r}{m};f\right],
\end{aligned}
$$

where the brackets represent the symbol for divided differences. If we use the identities

$$
\begin{aligned}
mx-k&=(m-r-k)\,x-k\,(1-x)+rx,\\
mx-r-k&=(m-r-k)\,x-k\,(1-x)-r\,(1-x),
\end{aligned}
$$

and take into account that

$$(m-r-k)\binom{m-r}{k}=(m-r)\binom{m-r-1}{k},\ k\binom{m-r}{k}=(m-r)\binom{m-r-1}{k-1},$$

we can write further

$$
\begin{aligned}
(R_{m,r}f)(x)=&\frac{m-r}{m}x\,(1-x)\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}x^k\,(1-x)^{m-r-k}\left[x,\frac{k}{m};f\right]+\\
&-\frac{m-r}{m}(1-x)^2\sum_{k=1}^{m-r}\binom{m-r-1}{k-1}x^k\,(1-x)^{m-r-k}\left[x,\frac{k}{m};f\right]+\\
&+\frac{r}{m}x\,(1-x)\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[x,\frac{k}{m};f\right]+\\
&+\frac{m-r}{m}x^2\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}x^k\,(1-x)^{m-r-k}\left[x,\frac{k+r}{m};f\right]-\\
&-\frac{m-r}{m}x\,(1-x)\sum_{k=1}^{m-r}\binom{m-r-1}{k-1}x^k\,(1-x)^{m-r-k}\left[x,\frac{k+r}{m};f\right]-\\
&-\frac{r}{m}x\,(1-x)\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[x,\frac{k+r}{m};f\right].
\end{aligned}
$$

Substituting $k-1=j$ in the second and fifth sums and then denoting again the summation index by k, we can write down the following expression for the remainder

$$(R_{m,r}f)(x)=\frac{m-r}{m}x(1-x)^2\sum_{k=0}^{m-r-1}p_{m-r-1,k}(x)\left\{\left[x,\frac{k}{m};f\right]-\left[x,\frac{k+1}{m};f\right]\right\}+$$

$$+\frac{r}{m}x(1-x)\sum_{k=0}^{m-r}p_{m-r,k}(x)\left\{\left[x,\frac{k}{m};f\right]-\left[x,\frac{k+r}{m};f\right]\right\}+$$

$$+\frac{m-r}{m}x^2(1-x)\sum_{k=0}^{m-r-1}p_{m-r-1,k}(x)\left\{\left[x,\frac{k+r}{m};f\right]-\left[x,\frac{k+r+1}{m};f\right]\right\}.$$

As a consequence of this result and of the following relations between divided differences

$$\left[x,\frac{k}{m};f\right]-\left[x,\frac{k+1}{m};f\right]=-\frac{1}{m}\left[x,\frac{k}{m},\frac{k+1}{m};f\right]$$

$$\left[x,\frac{k}{m};f\right]-\left[x,\frac{k+r}{m};f\right]=-\frac{r}{m}\left[x,\frac{k}{m},\frac{k+r}{m};f\right]$$

$$\left[x,\frac{k+r}{m};f\right]-\left[x,\frac{k+r+1}{m};f\right]=-\frac{1}{m}\left[x,\frac{k+r}{m},\frac{k+r+1}{m};f\right],$$

we obtain finally the representation (3.5) and so the Theorem 3.1 is proved.

One observes that for $r=0$ formula (3.5) leads us to the following expression for the remainder in the classical Bernstein approximation formula

$$(3.6)\qquad (R_{m,0}f)(x)=(R_mf)(x)=-\frac{x(1-x)}{m}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[x,\frac{k}{m},\frac{k+1}{m};f\right],$$

which has been established first in our earlier paper [17].

It is easily verified that in the case $r=1$ one obtains also formula (3.6), so that we have $R_{m,0}f=R_{m,1}f=R_mf$.

Formula (3.5) enables us to state the following corollaries.

COROLLARY 3.1. *We have* $(R_{m,r}f)(0)=(R_{m,r}f)(1)=0$, *and* $R_{m,r}f=0$*-if and only if* f *is a linear function.*

COROLLARY 3.2. *If* f *is convex of first-order on* $[0,1]$, *without being linear, then we have* $L_{m,r}f>f$ *on* $(0,1)$, *while if* f *is concave of first-order on* $[0,1]$ *then* $L_{m,r}f<f$ *on* (0.1).

COROLLARY 3.3. *If all the divided differences of second-order of* f *are bounded on* $[0,1]$, *then we have*

$$|(R_{m,r}f)(x)| \leq \left[1+\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m} M_2(f), \tag{3.7}$$

where $M_2(f)$ is the least upper bound of the absolute values of the second-order divided differences of f on $[0, 1]$.

The inequality (3.7) permits us to see at once that we can optimize the error bound of the approximation of the function f by means of $L_{m,r}f$ if we take $r=0$ or $r=1$, when the operator $L_{m,r}$ reduces to the Bernstein operator B_m.

COROLLARY 3.4. *If $f \in C[0, 1]$ then for any fixed point x of $[0, 1]$ we have*

$$(R_{m,r}f)(x) = -\left[1+\frac{r(r-1}{m}\right] \cdot \frac{x(1-x)}{m} [\xi_{m1}, \xi_{m2}, \xi_{m3}; f], \tag{3.8}$$

where ξ_{mj} $(j=1, 2, 3)$ are certain distinct points of $[0, 1]$, which might depend upon f.

PROOF. According to Theorem 3.1 the approximation formula (3.4) has the degree of exactness one and it is obvious that $R_{m,r}f \neq 0$ if f is any first-order convex function, i. e. a function for which any divided differences on three distinct points of $[0, 1]$ is positive. By using a known theorem of T. Popoviciu [13] there exist three distinct points $\xi_{m1}, \xi_{m2}, \xi_{m3}$ on $[0, 1]$ such that

$$(R_{m,r}f)(x) = (R_{m,r}e_2)(x) [\xi_{m1}, \xi_{m2}, \xi_{m3}; f].$$

But from the formula (2.3), where we set $\alpha=\beta=0$, we obtain

$$(L_{m,r}e_2)(x) = x^2 + \left[1+\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m}, \tag{3.9'}$$

so that we have

$$(R_{m,r}e_2)(x) = -\left[1+\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m} \tag{3.9}$$

and this completes the proof.

As a consequence of Corollary 3.4 we can state

COROLLARY 3.5. *If $f \in C^2[0, 1]$ then there exists a point $\xi_m \in (0, 1)$ such that*

$$(R_{m,r}f)(x) = -\left[1+\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{2m} f''(\xi_m). \tag{3.10}$$

Now we can give an asymptotic estimate, of Voronovskaja type, for the remainder of approximation formula (3.4).

THEOREM 3.2. *If the function* $f \in C[0,1]$ *possesses a second derivative at a point* x *of* $[0,1]$, *then we have*

$$(R_{m,r} f)(x) = -\left[1+\frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{2m} f''(x) + \frac{\varepsilon_m(x)}{m},$$

where $\varepsilon_m(x)$ *tends to* 0 *when* m *tends to* ∞.

We sketch the proof of this theorem. The idea here is to use Taylor expansion of the following form

$$f(t) = f(x) + (t-x) f'(x) + \frac{1}{2}(t-x)^2 [f''(x) + g(t)],$$

where g is a certain real-valued function defined on $[0,1]$ having the property: $g(t) \to 0$ when $t \to x$. Next we have to replace t by k/m, multiply both members by $w_{m,k,r}(x)$, sum over k and take into account that

$$L_{m,r} e_0 = e_0,\ L_{m,r} e_1 = e_1,\ L_{m,r} e_2 = e_2 + \left[1+\frac{r(r-1)}{m}\right] \frac{e_1-e_2}{m}.$$

Consequently, we obtain

$$(R_{m,r} f)(x) = -\left[1+\frac{r(r-1)}{m}\right] \frac{x(1-x)}{2m} f''(x) + \rho_m(x),$$

where

$$\rho_m(x) = \frac{1}{2} \sum_{k=0}^{m} w_{m,k,r}(x) \left(\frac{k}{m}-x\right)^2 g\left(\frac{k}{m}\right).$$

By using a standard method (see, e. g. [7]), it can be shown that $\rho_m(x)$ can be represented in the form $\varepsilon_m(x)/m$, where $\varepsilon_m(x) \to 0$ as $m \to \infty$.

We next proceed to establish an integral representation of the remainder of the approximation formula (3.4).

THEOREM 3.3. *If* $f \in C^2[0,1]$ *and* x *is any fixed point of* $[0,1]$ *then we have*

$$(R_{m,r} f)(x) = \int_0^1 G_{m,r}(t; x) f''(t)\, dt, \tag{3.11}$$

where

$$G_{m,r}(t;x)=(R_{m,r}\varphi_x)(t),\quad \varphi_x(t)=(x-t)_+=\frac{x-t+|x-t|}{2}$$

and $R_{m,r}$ operates on $\varphi_x(t)$ as a function of x.

Proof. Appealing to a well known theorem of G. Peano [11], we obtain formula (3.11), $G_{m,r}(\cdot\,;x)$ being the Peano kernel associated with the operator $R_{m,r}$.

Because

$$(R_{m,r}\varphi_x)(t)=(x-t)_+-\sum_{k=0}^{m} w_{m,k,r}(x)\left(\frac{k}{m}-t\right)_+,$$

it is easy to obtain explicit expressions for this kernel.

Assuming that

$$x\in\left[\frac{p-1}{m},\frac{p}{m}\right]\quad (1\le p\le m),$$

we have

$$\text{(i)}\quad G_{m,r}(t;x)=x-t-\sum_{k=j}^{m} w_{m,k,r}(x)\left(\frac{k}{m}-t\right)$$

for

$$t\in\left[\frac{j-1}{m},\frac{j}{m}\right],\quad j=1,2,\dots,p-1;$$

$$\text{(ii)}\quad G_{m,r}(t;x)=x-t-\sum_{k=p}^{m} w_{m,k,r}(x)\left(\frac{k}{m}-t\right),\quad t\in\left[\frac{p-1}{m},x\right];$$

$$\text{(iii)}\quad G_{m,r}(t;x)=-\sum_{k=p}^{m} w_{m,k,r}(x)\left(\frac{k}{m}-t\right),\quad t\in\left[x,\frac{p}{m}\right];$$

$$\text{(iv)}\quad G_{m,r}(t;x)=-\sum_{k=j}^{m} w_{m,k,r}(x)\left(\frac{k}{m}-t\right)$$

for

$$t\in\left[\frac{j-1}{m},\frac{j}{m}\right],\quad j=p+1,\,p+2,\dots,m.$$

Now referring to the identities

$$\sum_{k=0}^{j-1} w_{m,k,r}(x)+\sum_{k=j}^{m} w_{m,k,r}(x)=1,$$

$$\sum_{k=0}^{j-1} k w_{m,k,r}(x)+\sum_{k=j}^{m} k w_{m,k,r}(x)=mx,$$

it is easy to see that we can write

$$G_{m,r}(t;\,x)=-\sum_{k=0}^{j-1} w_{m,k,r}(x)\left(t-\frac{k}{m}\right) \tag{3.12}$$

for

$$t\in\left[\frac{j-1}{m},\,\frac{j}{m}\right]\qquad j=1,2,\ldots,p-1,$$

while for $t\in\left[\frac{p-1}{m},\,x\right]$ we obtain

$$G_{m,r}(t;\,x)=-\sum_{k=0}^{p-1} w_{m,k,r}(x)\left(t-\frac{k}{m}\right). \tag{3.13}$$

If we take into account (iii), (iv), (iv), (3.12) and (3.13), we see that for a fixed value $x\in[0,1]$ the equation $y=G_{m,r}(t;\,x)$ represents a continuous broken line which joins the points (0,0), (0,1) and is situated beneath the t-axis.

By application of the mean value theorem to the integral occurring in (3.11) we can obtain in another way formula (3.10), because

$$\int_0^1 G_{m,r}(t;\,x)\,dt=\frac{1}{2}\,(R_{m,r}\,e_2)(x)=-\left[\,1+\frac{r\,(r-1)}{m}\right]\frac{x\,(1-x)}{2m}.$$

Employing a method of Radon-Ghizzetti (cf. [14], [5] or [6]), it may actually be shown that $G_{m,r}(t;\,x)$ represents the solution of a second-order differential system, under certain boundary conditions, so that $G_{m,r}(t;\,x)$ is the corresponding Green's function. It should be pointed out that in fact $G_{m,r}(t;\,x)$ represents a spline function, of first degree, using the knots k/m.

One observes that the operator $L_{m,r}$ enjoys the variation diminishing property (see I. J. Schoenberg [15]), because according to Theorem 3.1 it preserves the linear functions and it is obvious that the number of variations of sign of $L_{m,r}\,f$ and of f on [0, 1] satisfies the relation: $\nu\,(L_{m,r}\,f)\leq\nu\,(f)$.

4. Estimate of the order of approximation.

In this section we are concerned with the estimate of the order of approximation of a function $f \in C\,[0,1]$ by means of the linear positive operator $L_{m,r}$. We shall use the modulus of continuity, defined by

$$\omega(\delta) = \omega(f;\delta) := \sup |f(x'') - f(x')|,$$

where x' and x'' are points from $[0,1]$ so that $|x''-x'| < \delta$, δ being a positive number.

By using a standard method we prove

THEOREM 4.1. *If $f \in C\,[0,1]$ then for any $x \in [0,1]$ we have*

$$|f(x) - (L_{m,r}f)(x)| \le \left[1 + \frac{1}{\alpha}\sqrt{1 + \frac{r(r-1)}{m}}\right] \omega\left(\alpha\sqrt{\frac{x(1-x)}{m}}\right). \tag{4.1}$$

where α is any positive constant.

PROOF. Since $L_{m,r}e_0 = e_0$ and $w_{m,k,r}(x) \ge 0$ on $[0,1]$, we can write

$$|f(x) - (L_{m,r}f)(x)| \le \sum_{k=0}^{m} w_{m,k,r}(x) \left| f(x) - f\left(\frac{k}{m}\right)\right|.$$

If we use the following known properties of the modulus of continuity

$$|f(x'') - f(x')| \le \omega(|x''-x'|), \quad \omega(\lambda\delta) \le (1+\lambda)\,\omega(\delta),$$

we obtain

$$\left| f(x) - f\left(\frac{k}{m}\right)\right| \le \omega\left(\frac{1}{\delta}\left|x - \frac{k}{m}\right|\delta\right) \le \left(1 + \frac{1}{\delta}\left|x - \frac{k}{m}\right|\right)\omega(\delta).$$

Consequently we can write

$$|f(x) - (L_{m,r}f)(x)| \le \left(1 + \frac{1}{\delta}\sum_{k=0}^{m} w_{m,k,r}(x)\left|x - \frac{k}{m}\right|\right)\omega(\delta). \tag{4.2}$$

By making use of the Cauchy inequality we obtain

$$\sum_{k=0}^{m} w_{m,k,r}(x)\left|x - \frac{k}{m}\right| \le \left[\sum_{k=0}^{m} w_{m,k,r}(x)\left(x - \frac{k}{m}\right)^2\right]^{1/2} = \tag{4.3}$$

$$= [(L_{m,r}e_2)(x) - x^2]^{1/2} = \left\{\left[1 + \frac{r(r-1)}{m}\right]\frac{x(1-x)}{m}\right\}^{1/2}$$

because of the equality (3.9'). Hence

$$|f(x)-(L_{m,r}f)(x)|\leq\left\{1+\frac{1}{\delta}\left[\left(1+\frac{r(r-1)}{m}\right)\frac{x(1-x)}{m}\right]^{1/2}\right\}\omega(\delta).$$

By inserting into it $\delta=\alpha\sqrt{x(1-x)}/\sqrt{m}$, where α denotes any positive constant (naturally with the condition that this value of δ belongs to the domain of ω), we are led just to the desired inequality (4.1).

In the special cases $r=0$ or $r=1$, we obtain the corresponding inequality for the Bernstein operators:

$$|f(x)-(B_m f)(x)|\leq\left(1+\frac{1}{\alpha}\right)\omega\left(\alpha\sqrt{\frac{x(1-x)}{m}}\right).$$

Since $x(1-x)\leq 1/4$, if we take $\alpha=2$, we obtain the following result.

COROLLARY 4.1. *In the maximum norm over* $[0,1]$ *we have*

$$\|f-L_{m,r}f\|\leq\left(1+\frac{1}{2}\sqrt{1+\frac{r(r-1)}{m}}\right)\omega\left(\frac{1}{\sqrt{m}}\right). \tag{4.4}$$

If we next assume that the function f possesses a continuous derivative on the interval $[0,1]$, then we may proceed to give a new estimate of the order of approximation of f by means of $L_{m,r}f$.

THEOREM 4.2. *If* $f\in C^1[0,1]$ *then we have*

$$|f(x)-(L_{m,r}f)(x)|\leq \tag{4.5}$$

$$\sqrt{1+\frac{r(r-1)}{m}}\left(1+\frac{1}{\alpha}\sqrt{1+\frac{r(r-1)}{m}}\right)\sqrt{\frac{x(1-x)}{m}}\,\omega\left(f';\alpha\sqrt{\frac{x(1-x)}{m}}\right),$$

α being any positive constant.

PROOF. By applying the mean value theorem of differential calculus we can write

$$f(x)-f\left(\frac{k}{m}\right)=\left(x-\frac{k}{m}\right)f'(x)+\left(x-\frac{k}{m}\right)[f'(\xi)-f'(x)],$$

where $\xi=\xi_{m,k}(x)$ is an interior point of the interval determined by x and k/m.

If we multiply both members of this equality by $w_{m,k,r}(x)$ and sum over *k*, we obtain

$$f(x)-(L_{m,r}f)(x)=f'(x)\sum_{k=0}^{m}\left(x-\frac{k}{m}\right)w_{m,k,r}(x)+$$

$$+\sum_{k=0}^{m}\left(x-\frac{k}{m}\right)w_{m,k,r}(x)\,[f'(\xi)-f'(x)].$$

Since $L_{m,r}\,e_j=e_j$ $(j=0, 1)$, the first sum vanishes and we can write

$$|f(x)-(L_{m,r}f)(x)|\leq\sum_{k=0}^{m}\left|x-\frac{k}{m}\right|w_{m,k,r}(x)\,|f'(\xi)-f'(x)|.$$

One observes that we have

$$|f'(\xi)-f'(x)|\leq\left(1+\frac{1}{\delta}\,|\xi-x|\right)\omega(f';\delta)\leq\left(1+\frac{1}{\delta}\left|\frac{k}{m}-x\right|\right)\omega(f';\delta),$$

where δ is any positive number which does not depend on *k*.

Consequently we get

$$|f(x)-(L_{m,r}f)(x)|\leq\left[\sum_{k=0}^{m}\left|x-\frac{k}{m}\right|w_{m,k,r}(x)+\right.$$

$$\left.+\frac{1}{\delta}\sum_{k=0}^{m}\left(x-\frac{k}{m}\right)^2 w_{m,k,r}(x)\right]\cdot\omega(f';\delta).$$

By using the relationship (4.3) and inserting $\delta=\alpha\,x(1-x)/\,m$, we obtain finally the desired inequality.

As in the case of the Corollary 4.1, we can deduce from Theorem 4.2 the following

COROLLARY 4.2. *If* $f\in C^1[0, 1]$, *we have in the maximum norm over* $[0, 1]$:

$$(4.6)\quad \|f-L_{m,r}f\|\leq\frac{1}{4\sqrt{m}}\sqrt{1+\frac{r(r-1)}{m}}\left[2+\sqrt{1+\frac{r(r-1)}{m}}\right]\omega\left(f';\frac{1}{\sqrt{m}}\right).$$

One observes that for $x=0$ or $x=1$ the relations (4.1) and (4.5) become equalities because of (3.3).

It should be noticed that in the special cases $r=0$ or $r=1$ the inequalities (4.4) and (4.6) reduce respectively to the classical inequalities of T. Popoviciu and G. G. Lorentz.

Remark. In 1969 [19] we have evaluated the orders of approximation of functions by means of a general class of linear positive operators (L_m) constructed by a probabilistic method. Our results were formulated in terms of modulus of continuity of the function f or of its derivative f', as well as of an arbitrary positive constant which can be properly chosen in every case. This idea to introduce and use such a constant has been adopted, unfortunately ignoring the real source [19], by B. Mond [9] and B. Mond and R. Vasudevan [10].

5. The spectrum of the operator $L_{m,r}$ and a quadrature formula.

We now proceed to examine the spectral properties of the operator $L_{m,r}$ mapping the Banach space $C[0, 1]$ into itself.

It is known that a number λ such that $L_{m,r}f=\lambda f$ for some element $f\in C[0, 1]$, which is not identically zero, is called an eigenvalue of $L_{m,r}$, while f is an eigenfunction.

According to (1.7), if $r\geq 1$ then for each $j=0, 1, \dots, m-r+1$ the operator $L_{m,r}$ transforms a polynomial P_j, of degree j, into a polynomial of the same degree; consequently for each j there corresponds an eigenfunction associated with each eigenvalue $\lambda_j=\lambda_j(L_{m,r})$. By using the method of undetermined coefficients the equation $L_{m,r}P_j=\lambda_j P_j$ permits to determine the eigenvalues λ_j and the corresponding eigenfunctions, that is the spectrum of the operator $\mathrm{L}_{m,r}$.

We have obtained the following result regarding the point spectrum of the operator $L_{m,r}$.

Theorem 5.1. *The eigenvalues $\lambda_j(m, r)$ of the operator $L_{m,r}$ are given by*

$$\lambda_0(m, r)=\lambda_1(m, r)=1,$$

$$\lambda_j(m, r)=\left(1-\frac{r}{m}\right)\left(1-\frac{r+1}{m}\right)\dots\left(1-\frac{r+j-2}{m}\right)\left(1+\frac{(j-1)(r-1)}{m}\right)$$

$$(2\leq j\leq m-r+i).$$

If we set here $r=1$ we see that the eigenvalues $\lambda_j(m, 0)=\lambda_j(m)$ of the Bernstein operator B_m are

$$\lambda_0(m)=\lambda_1(m)=1, \quad \lambda_j(m)=\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{j-1}{m}\right) \quad (2\leq j\leq m).$$

Note that the point spectrum of the operator B_m was first given in [3]. Recently in [1] it has beeen given a characterization of the operator B_m by using the eigenvalue $\lambda_2(m)$.

It is easily verified that in our case we have

$$0<\lambda_{m-r+1}(m, r)<\ldots<\lambda_2(m, r)<\lambda_1(m, r)=\lambda_0(m, r)=1.$$

As we have seen at (4.1), the rate of convergence of the operator $L_{m,r}$ is characterized by $L_{m,r}((t-x)^2; x)$. Since

$$\frac{x(1-x)}{m}=B_m((t-x)^2; x)\leq L_{m,r}((t-x)^2; x)=\left[1+\frac{r(r-1)}{m}\right]\frac{x(1-x)}{m},$$

it is clear that the best result can be achieved when $r=0$ or $r=1$, that is in the case of the operator B_m.

Because of the fact that

$$1-\frac{1}{m}-\frac{r(r-1)}{m^2}=\lambda_2(L_{m,r})\leq\lambda_2(B_m)=1-\frac{1}{m},$$

we are able to arrive at the same conclusion by using a theorem given in [1].

We mention that to each eigenvalue $\lambda_j(m, r)$ there corresponds an infinite number of polynomials which are eigenfunctions associated with this eigenvalue. Thus, for the particular degrees 0, 1, 2, it is easy to see that the corresponding eigenfunctions are α, $\alpha+\beta x$, $\alpha x(1-x)$, where α and β are arbitrary real numbers.

We end this paper pointing out that by using the linear positive operator $L_{m,r}$ we have constructed (see our recent paper [20]) the following quadrature formula

$$\int_0^1 f(x)\,dx=\frac{1}{(m-r+1)(m-r+2)}\left[\sum_{k=0}^{r-1}(m-r-k+1)\,f\left(\frac{k}{m}\right)+\right.$$

$$\left.+(m-2r+2)\sum_{k=r}^{m-r}f\left(\frac{k}{m}\right)+\sum_{k=m-r+1}^{m}(k-r+1)\,f\left(\frac{k}{m}\right)\right]+\rho_{m,r}(f),$$

where, if we assume that $f\in C^2(0, 1)$, the remainder can be expressed in the following simple form

$$\rho_{m,r}(f)=-\frac{1}{12m}\left[1+\frac{r(r-1)}{m}\right]f''(\xi),\quad 0<\xi<1.$$

In the case $L_{m,0}=L_{m,1}=B_m$ this result has been given in our earlier paper [16].

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