sàro operators, Acta Sci.
s, Cambridge Univ. Press,
Acta Sci. Math. (Szeged),
atica, **22**, (1980) 97 – 105.
a Sci. Math. (Szeged), 32,

ndiana University
Bloomington IN 47405 *SUA*

MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

MATHEMATICA, Tome 26 (49), N° 2, 1984, pp. 153 – 157

Dedicated to Professor
I. J. SCHOENBERG
on his 80th birthday

A NOTE ON A MULTIPARAMETER BERNSTEIN-TYPE APPROXIMATING OPERATOR

by

D. D. STANCU

In a previous paper [4] we have constructed and investigated a positive linear operator, of Bernstein-type, depending on a non-negative integer parameter r and on two real parameters α and β, such that: $0 \leqslant \alpha \leqslant \beta$.

The purpose of this note is to consider and study a more general operator $L^{\alpha,\beta}_{m,r_1,\ldots,r_s}$ mapping into itself the Banach space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$. This operator depends on the parameters α and β, as well as on s non-negative integer parameters $r_1, \ldots, r_s$, independent of the natural number m, and such that: $0 \leqslant r_1 \leqslant \ldots \leqslant r_s$, $r_1 + \ldots + r_s < m$.

For any function $f\colon [0, 1] \to R$, we define this operator by

$$(1)\qquad (L^{\alpha,\beta}_{m,r_1,\ldots,r_s} f)(x) := \sum_{k=0}^{m-r_1-\ldots-r_s} p_{m-r_1\ldots-r_s,k}(x) F^{r_1,\ldots,r_s}_{m,k}(f;\ \alpha,\ \beta;\ x),$$

where

$$(2)\qquad F^{r_1,\ldots,r_s}_{m,k}(f;\ \alpha,\ \beta;\ x) = \sum_{j=0}^{s} x^j (1-x)^{s-j} G^{r_1,\ldots,r_s}_{m,k,j}(f;\ \alpha,\ \beta),$$

with

$$(3)\qquad \begin{aligned} G^{r_1,\ldots,r_s}_{m,k,j}(f;\ \alpha,\ \beta) &= f\left(\frac{k + r_1 + \ldots + r_j + \alpha}{m+\beta}\right) + f\left(\frac{k + r_2 + \ldots + r_{j+1} + \alpha}{m+\beta}\right) + \\ &+ f\left(\frac{k + r_1 + r_3 + \ldots + r_{j+1} + \alpha}{m+\beta}\right) + \ldots + f\left(\frac{k + r_{s-j+1} + \ldots + r_{s-1} + r_s + \alpha}{m+\beta}\right) \end{aligned}$$

and with a usual notation, which will be used throughout the paper:

$$(4)\qquad p_{n,i}(t) := \binom{n}{i} t^i (1-t)^{n-i}.$$

In the first place, one observe that we have

$$(5)\qquad (L^{\alpha,\beta}_{m,r_1,\ldots,r_s}f)(0) = f\left(\frac{\alpha}{m+\beta}\right), \qquad (L^{\alpha,\beta}_{m,r_1,\ldots,r_s}f)(1) = f\left(\frac{m+\alpha}{m+\beta}\right).$$

Consequently, the polynomial defined at (1) is interpolatory at the left side of [0, 1] if $\alpha = 0$ and, respectively, at the right side if $\beta = 0$. For $\alpha = \beta = 0$ it is interpolatory at both sides of [0, 1].

In the special case: $r_1 = \ldots = r_s = r$ we obtain

$$(6)\qquad (L^{\alpha,\beta}_{m,r,s}f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \sum_{j=0}^{s} p_{s,j}(x) f\left(\frac{k+\alpha+jr}{m+\beta}\right),$$

which for $s = 0$ leads us to the Bernstein-type operator $B^{\alpha,\beta}_m$ investigated in our earlier paper [3], while for $s = 1$ we arrive at the operator $L^{\alpha,\beta}_{m,r}$ studied in detail in our recent paper [4]. If $\alpha = \beta = 0$, $s = 1$ and $r = 0$ or $r = 1$, we get the well known Bernstein operatpor B_m.

A convergence property of the sequence of approximating polynomials defined at (1) is illustrated by the following theorem.

THEOREM 1. *If* $f \in C[0, 1]$, $r_1, \ldots, r_s$ *are fixed non-negative integers and* $0 \leqslant \alpha \leqslant \beta$, *then we have*

$$\lim_{m\to\infty} L^{\alpha,\beta}_{m,r_1,\ldots,r_s} f = f,$$

uniformly on the interval [0, 1].

Because we have a sequence of linear positive operators mapping into itself the space $C[0, 1]$, we need first to find the values of the operator for the three test functions e_0, e_1, e_2, where $e_j(t) := t^j$ $(j = 0, 1, 2)$ for any $t \in [0, 1]$.

It is a straightforward calculation to verify that we have

$$(7)\qquad (L^{\alpha,\beta}_{m,r_1,\ldots,r_s}e_0)(x) = 1, \quad (L^{\alpha,\beta}_{m,r_1,\ldots,r_s}e_1)(x) = x + \frac{\alpha-\beta x}{m+\beta},$$

$$(L^{\alpha,\beta}_{r_1,\ldots,r_s}e_2)(x) = x^2 + (1 + S^{r_1,\ldots,r_s}_{m,}) \cdot \frac{x(1-x)}{m+\beta} + \frac{(\alpha-\beta x)(\alpha+\beta x+2mx)}{(m+\beta)^2},$$

where, for abbreviation, we have introduced the notation

$$(8)\qquad S^{r_1,\ldots,r_s}_{m,\beta} = \frac{1}{m+\beta}\left[\sum_{j=1}^{s} r_j(r_j-1) - \beta\right].$$

These results permit us to see that the uniform convergence holds for the monomials e_0, e_1 and e_2, since the quantity (8) tends to zero when m tends to infinity. The assertion of the theorem now follows immediately by applying the well known convergence criterion of Bohman—Korovkin.

One observes that if $\alpha = \beta = 0$ then e_0 and e_1 are fixed elements of the operator $L_{m,r_1,\ldots,r_s} = L^{0,0}_{m,r_1,\ldots,r_s}$ and the approximation formula

$$(9)\qquad f = L_{m,r_1,\ldots,r_s}f + R_{m,r_1,\ldots,r_s}f$$

has the highest degree of exactness, namely $N = 1$.

We shall henceforth restrict to this important special case, when we can write

$$(L_{m,r_1,\ldots,r_s}f)(x) = \sum_{k=0}^{m-r_1-\ldots-r_s} p_{m-r_1-\ldots-r_s,k}(x)F_{m,k}^{r_1,\ldots,r_s}(f;\ x), \tag{10}$$

and, more explicitely, we now have

$$F_{m,k}^{r_1,\ldots,r_s}(f;\ x) = (1-x)^s f\left(\frac{k}{m}\right) + x(1-x)^{s-1}\left[f\left(\frac{k+r_1}{m}\right) + \ldots + f\left(\frac{k+r_s}{m}\right)\right] +$$

$$+ x^2(1-x)^{s-2}\left[f\left(\frac{k+r_1+r_2}{m}\right) + f\left(\frac{k+r_2+r_3}{m}\right) + f\left(\frac{k+r_1+r_3}{m}\right) + \ldots + f\left(\frac{k+r_{s-1}+r_s}{m}\right)\right] +$$

$$+ \ldots + x^{s-1}(1-x)\left[f\left(\frac{k+r_1+\ldots+r_s}{m}\right) + f\left(\frac{k+r_1+r_3+\ldots+r_s}{m}\right) + \ldots +\right.$$

$$\left. + f\left(\frac{k+r_1+\ldots+r_{s-1}}{m}\right)\right] + x^s f\left(\frac{k+r_1+\ldots+r_s}{m}\right).$$

According to (7), we now have

$$(R_{m,r_1,\ldots,r_s}e_2)(x) = -\frac{x(1-x)}{m}\left(1 + S_{m,0}^{r_1,\ldots,r_s}\right). \tag{11}$$

In the particular case of the operator defined at (6) we obtain

$$(R_{m,r,s}\ e_2)(x) = -\frac{x(1-x)}{m}\left[1 + \frac{sr(r-1)}{m}\right].$$

By using a standard method one can prove

THEOREM 2. *If $f \in C[0,\ 1]$ then for any $x \in [0,\ 1]$ we have*

$$|f(x) - (L_{m,r_1,\ldots,r_s}f)(x)| \leqslant$$

$$\leqslant \left[1 + \frac{1}{\gamma}\sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\right]\omega\left(f;\ \gamma\sqrt{\frac{x(1-x)}{m}}\right),$$

where ω is the modulus of continuity, while γ is a unspecified positive constant.

This theorem enables us to see that in the maximum norm over $[0,\ 1]$ we can deduce as a corrollary of this theorem the following estimate of T. Popoviciu — type:

$$||f - L_{m,r_1,\ldots,r_s}f|| \leqslant \left(1 + \frac{1}{2}\sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\right)\omega\left(f;\ \frac{1}{\sqrt{m}}\right).$$

In the next theorem we indicate a result from which there follows at once an estimate of G. G. Lorentz—type.

THEOREM 3. *If $f \in C^1[0,\ 1]$ then we have*

$$|f(x) - (L_{m,r_1,\ldots,r_s}f)(x)| \leqslant$$

$$\leqslant \sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\left(1 + \frac{1}{\gamma}\sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\right)\sqrt{\frac{x(1-x)}{m}}\,\omega\left(f';\ \gamma\sqrt{\frac{x(1-x)}{m}}\right).$$

The following theorem gives an evaluation of the remainder of the approximation formula (9).

THEOREM 4. *If $f \in C^2[0, 1]$ and x is any given point of $[0, 1]$ then we have the following integral representation*

$$(12) \qquad (R_{m,r_1,\ldots,r_s}f)(x) = \int_0^1 G_{m,r_1,\ldots,r_s}(t;\ x) f''(t)dt,$$

where

$$(13) \qquad G_{m,r_1,\ldots,r_s}(t;\ x) = (R_{m,r_1,\ldots,r_s}\varphi_x)(t), \quad \varphi_x(t) = (x - t)_+$$

and $R_{m,r_1,\ldots,r_s}$ operates on $\varphi_x(t)$ as a function of x.

Since the degree of exactness of formula (9) is $N = 1$, it is clear that by using a well known theorem of Peano the remainder can be represented under the form (12), at (13) being defined the Peano kernel associated with our operator. One can see that for a fixed value of x the equation $y = G_{m,r_1,\ldots,r_s}(t;\ x)$ represents a continuous broken line which joins the points (0, 0) and (0, 1), being situated beneath the t-axis. One observes that it represents a spline function, of first degree, having the knots k/m.

By use of the mean value theorem we obtain

$$(14) \qquad (R_{m,r_1,\ldots,r_s}f)(x) = f''(\xi) \int_0^1 G_{m,r_1,\ldots,r_s}(t;\ x)dt \quad (0 < \xi < 1).$$

Because the Peano kernel is independent of the function f, we may insert $f = e_2$ in (14) and we obtain

$$(15) \qquad \int_0^1 G_{m,r_1,\ldots,r_s}(t;\ x)dt = \frac{1}{2}(R_{m,r_1,\ldots,r_s}e_2)(x).$$

Consequently, from (14), (15) and (11) we can deduce a Cauchy-type expression for the remainder:

$$(R_{m,r_1,\ldots,r_s}f)(x) = -\frac{x(1-x)}{2m}[1 + S_{m,0}^{r_1,\ldots,r_s}]f''(\xi).$$

It should be observed that if f is convex of first-order on [0, 1] then we have $L_{m,r_1,\ldots,r_s}f > f$ on [0, 1], while if f is concave of first-order on [0, 1] then we have $L_{m,r_1,\ldots,r_s}f < f$ on (0, 1).

By using a standard technique, one can give an asymptotic estimate, of Voronovskaja-type, for the remainder of the approximation formula (9); we can state

THEOREM 5. *For any f in $C[0, 1]$, possessing a second derivative at a point x of [0, 1], we have*

$$(R_{m,r_1,\ldots,r_s}f)(x) = -[1 + S_{m,0}^{r_1,\ldots,r_s}]\frac{x(1-x)}{2m}f''(x) + \frac{\varepsilon_m(x)}{m},$$

where $\varepsilon_m(x)$ tends to zero when m tends to infinity.

Now let us conclude this note by mentioning two other important properties of our operator : 1) This operator enjoys the variation diminishing property, in the sense of I. J. SCHOENBERG [2], since it preserves the linear functions and it is easy to see that the numbers of variations of sign of $L_{m,r_1,\ldots,r_s}f$ and f, on $[0, 1]$, satisfy the relation: $v(f) \geqslant v(L_{m,r_1,\ldots,r_s}f)$; 2) Assuming that $f' \in \mathrm{Lip}_M\, 1$, choosing $\gamma = 1$, and taking into account that in this case $\omega(f'; \delta) \leqslant M\delta$, we can deduce from Theorem 3 an estimate of the following form

$$|f(x) - (L_{m,r_1,\ldots,r_s}f)(x)| \leqslant M \cdot N_m^{r_1,\ldots,r_s} \cdot \frac{x(1-x)}{m},$$

where

$$N_m^{r_1,\ldots,r_s} = \sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\left[1 + \sqrt{1 + S_{m,0}^{r_1,\ldots,r_s}}\right].$$

This result gives an indication on the saturation class of the polynomial $L_{m,r_1,\ldots,r_s}f$, which consists of those $f \in C[0, 1]$ for which $f' \in \mathrm{Lip}\, 1$, the optimal order of approximation being $x(1-x)/m$. For proving these results one can make use of the techniques employed by G. G. LORENTZ [1] in the case of the classical Bernstein polynomial.

REFERENCES

[1] Lorentz, G. G., *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.

[2] Schoenberg, I. J., *On variation diminishing approximation methods*, in "On Numerical Approximation", 249–274 (ed. R. E. Langer), University of Wisconsin, Madison, 1959.

[3] Stancu, D. D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ Babeş-Bolyai, Math. — Phys., **14** (1969), 31–35.

[4] Stancu, D. D., *Approximation of functions by means of a new generalized Bernstein operator*, CALCOLO **20** (1983), 211–229.

Received 07.12.1982.

Facultatea de Matematică
Universitatea din Cluj-Napoca
3400 *Cluj-Napoca, România*