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A DUAL PROOF FOR THE LINEARIZATION OF THE CONVEXITY SPACES

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1. ABSTRACT. We consider a convexity space X in the sense of V.W.BRYANT and R.J.WEBSTER [1]. In [8] we have proved that for such a space the incidence and order HILBERT's axioms adapted for an arbitrary dimension are satisfied. Thus the axioms I-VIII of O.VEBLEN [10] hold. If dim $X \geq 3$ then the axiom IX also holds. The axiom $\underline{X}$ is equivalent with dim $X \leq 3$ and if X is complete then as a consequence the axiom XI holds.

Therefore if X is complete, of dimension 3 and satisfies the parallel postulate then all of VEBLEN's axioms (I-XII) hold. In this case it is well Known [10] that there exists a real linearization of X; this means that X can be organized as a real linear space such that its convexity structure be that indebted to the algebraic structure.

J.P.DOIGNON [5] and J.CANTWELL, D.C.KAY [4] have proved the existence of a real linearization for any complete convexity space of dimension ≥ 3 satisfying the parallel postulate; the linearization is unique up to a translation of the origin according to a theorem of D.C.KAY, W.MEYER [6].

In our paper this result will be proved from a different point of view using a more general result of P.MAH, S.A.NAIMPALLY, J.H.M.WHITFIELD [7].

We can summarize our proof as follows: First we solve the case $3 \leq n = \dim X < \infty$ by mathematical induction after n.

In the case dim $X = \infty$ we represent X by $Y \times L$, where $Y \subset X$ is a hyperplane of X and $L \subset X$ is a line meeting Y, identified with $\mathbb{R}$. We denote by Y^* the dual of Y consisting of all functions f defined on Y and with values in $\mathbb{R} \equiv L$ having the graph a hyperplane in $Y \times L = X$ and satisfying $f(\theta)=0$, where $\theta = Y \cap L$.

We shall prove that Y^* is a linearization family ([7]) of Y. Consequently we deduce the existence of a real linearization of Y and finally of $Y \times L = X$. To identify $L \equiv \mathbb{R}$, to prove that Y^* is a linearization family of Y and to show that the product of the linearization structures of Y and L is a linearization for $X = Y \times L$, we employ the result from the case $3 \leq \dim < \infty$ and the uniquence theorem from [6]. Thus our proof is essentially based on the papers: [10] for the case of dimension 3, [7] for a more general result on the linearization and [6] for the uniquence of the linearization.

2. INTRODUCTION. We denote by $\mathcal{L}(X)$ a linear structure $(F,+,\circ)$ over a field F on the set X and by $\mathcal{L}(X)|_Y$ the linear structure induced by $\mathcal{L}(X)$ on the subspace $Y \subset X$. If $\mathcal{L}(X')=(F,+',\circ')$ is a linear structure of another space X', then $\mathcal{L}(X) \times \mathcal{L}(X')$ means the linear structure $(F, \oplus, \odot)$ on $X \times X'$ defined by: $(x_1,x_1') \oplus (x_2,x_2') = (x_1+x_2, x_1'+'x_2')$

$$\lambda \odot (x,x') = (\lambda \circ x, \lambda \circ' x') .$$

A pair $(X, \mathcal{C})$ where X is a set and $\mathcal{C}$ is a family of subsets of X is called abstract convexity space (A.C.S.) if is closed under arbitrary intersections. The members of $\mathcal{C}$ are called convex sets and the convex hull of $A \subset X$ is defined by $[A]=\bigcap(C: A \subset C,\ C \in \mathcal{C})$. Instead of the notation $[A \cup A']$ we use $[A, A']$. For $x,y \in X$ $[x,y]$ is named the closed segment joining x and y; the open segment, denoted by $x \cdot y$ is defined by

$$(1) \qquad x \cdot y = \begin{cases} [x,y] \setminus (x,y) & , \text{ if } x \neq y \\ x & , \text{ if } x = y \end{cases}$$

The open half-line having the origin in x and not containing y, denoted x/y, is defined by

$$(2) \qquad x/y = (z: x \in z \cdot y)$$

and the line containing x,y , $x \neq y$ is the set

$$(3) \qquad \{x,y\} = x/y \cup x \cup x \cdot y \cup y \cup y/x .$$

A set $A \subset X$ is said to be linear if $\{x,y\} \subset A$ for every $x,y \in A$. On an A.C.S. $(X, \mathcal{C})$ we formulate the following three conditions:

$$(4) \qquad [A]=\bigcup([B]: B \subset A,\ \text{card } B < \infty) \text{ for each } A \subset X$$

$$(5) \qquad [x,A]=\bigcup([x,a]: a \in [A]) \text{ for } A \subset X,\ \text{card } A < \infty, x \in X$$

$$(6) \qquad [x,y]=[z,y] \Rightarrow x = z .$$

It is important to remarck that if $(X, \mathcal{C})$ is an A.C.S. satisfying (4) and (5) then

$$(7) \qquad C \in \mathcal{C} \Longleftrightarrow x \cdot y \subset C \quad (\forall)\ x,y \in C .$$

The determination of a linear structure $(F,+,\circ)$ for $(X, \mathcal{E})$ which makes X a vector space over an ordered field F whose algebraic convex sets are presisely the members of $\mathcal{E}$ has been called the <u>linearization problem</u> for an A,C,S, In the case $F = \mathbb{R}$ we speak about a <u>real linearization</u>.

If $\mathcal{L}(X)=(\mathbb{R},+,\circ)$ is a real linear structure for X such that the open segment idebted to $\mathcal{E}$ defined by (1) is the same with the open segment idebted to $\mathcal{L}(X)$ we say that $\mathcal{L}(X)$ is a <u>strong real linearization</u> of X.

Clearly if $\mathcal{L}(X)$ is a strong real linearization of X then the open half-line indebted to $\mathcal{E}$ is identical with that indebted to $\mathcal{L}(X)$ and so by (3) it follows that $A \subset X$ is a linear set in $(X, \mathcal{E})$ if and anly if it is an algebraic linear set.

Moreover, if $(X, \mathcal{E})$ satisfies (4) and (5) then by (7) the notions of real linearization and strong linear linearization for X are identical.

To obtain a characterization of the A.C.S. having a real linearization, a family X^* of real functions defined on X is called in [7] a <u>linearization family</u> for X provided that the following four conditions are satisfied :

(8) if $f \in X^*$, $C \in \mathcal{E}$ then $f(C)$ is convex in $\mathbb{R}$,

(9) there exists $x_0 \in X$ with $f(x_0)=0$ for each $f \in X^*$, and if $f(x)=f(y)$ for each $f \in X^*$, then $x=y$.

(10) each $f \in X^*$ restricted to any line in X is either a bijection or a constant map.

(11) if $f,g \in X^*$ and each separates x and y, then there are $\lambda, \mu \in \mathbb{R}$ such that for each $z \in \{x,y\}$: $g(z)=\lambda f(z)+\mu$.

From [7] we know :

(A). <u>An abstract convexity space</u> $(X, \mathcal{E})$ <u>satisfying</u> (4), (5),(6) <u>has a strong real linearization if and only if</u> X <u>has a linearization family</u> X^* .

This characterization is an external one using the linearization family. The question is: how can an A.C.S. having a real linearization be characterized using only the properties of $\mathcal{E}$?

To answer at this question we formulate some conditions on the operations „•"(open segment) and „/"(open half-line) defined by (1) and (2) :

(12) $x \circ y \neq \emptyset$, $x/y \neq \emptyset$

(13) $x \circ x = x = x/x$

(14) $x \circ (y \circ z) = (x \circ y) \circ z$

(15) $x/y \cap x'/y' \neq \emptyset \Longrightarrow x \circ y' \cap y \circ x' \neq \emptyset$

(16) $x \circ y \cap x \circ z \neq \emptyset \Longrightarrow y = z$ or $y \in x \circ z$ or $z \in x \circ y$

($A \circ B$ stands for $\bigcup(a \circ b : a \in A, b \in B)$).

An abstract convexity space $(X, \mathcal{E})$ satisfying (12)-(16) is naimed in [1] convexity space. It will be denoted by $(X, \circ)$. For such a space (4),(5),(6) hold.

The notation, terminology and results from [1], [2], [3] will be used.

Let us mention the following theorem [6] which will be needed :

(B). If an A.C.S. admits a real linearization of dimension >1, then it is unique up to a translation of the origin.

3. THE MAIN RESULT. The aim of this paper is to do a new proof of the following :

THEOREM. If $(X, \circ)$ is a convexity space, complete, of dimension ≥ 3 satisfying the parallel postulate, then there exists a strong real linearization for X, unique up to a translation of origin.

This theorem can be reformulated in the following way:

An abstract convexity space $(X, \mathcal{E})$ has a strong real linearization of dimension ≥ 3 if and only if $\mathcal{E}$ satisfies (12)-(16), the parallel postulate and $(X, \circ)$ is complete and of dimension ≥ 3.

4. PROOF. Case 1. $3 \leq n = \dim X < \infty$. For $n = 3$ the theorem follows from [10]. For $n > 3$ we prove it by mathematical induction. Let us assume that it is true for n; we shall prove it for n+1. For, let $B = (\theta, e_1, e_2, \dots, e_{n+1})$ be a basis for X where $\dim X = n+1 \geq 4$. Define $Y = \{\theta, e_1, e_2, \dots, e_n\}$ and $L = \{\theta, e_{n+1}\}$. Y with the convexity structure induced by that of X becomes a complete convexity space of dimension n satisfying the parallel postulate. By the induction hypothesis it follows that Y admits an unique strong real linearization having the origin in θ, denoted by $\mathcal{L}(Y)$.

In the same way $Y_1 = \{\theta, e_2, \dots, e_{n+1}\}$ of dimension n has an unique strong real linearization with the origin in θ: $\mathcal{L}(Y_1)$. Since $L \subset Y_1$ it follows that $\mathcal{L}(Y_1)$ induces on L a strong real linearization $\mathcal{L}(L)$ having origin in θ .

Because X satisfies the parallel postulate it is possible the representation $X = Y \times L$; for we identify each $x \in X$ with the pair (y, ℓ) where y is the intersection of Y with the parallel through x to L and ℓ is the intersection of L with the parallel through x to the line $\{\theta, y\}$. If $x \in L$ then $\ell = x$. In particular each $x \in Y$ is identified with the pair (x, θ) and each $x \in L$ with (θ, x).

We have only to prove that $\mathcal{L}(Y) \times \mathcal{L}(L)$ is a strong linearization for $Y \times L = X$; that means that for every $x, x' \in X$, $x \neq x'$ the open segment having its ends in x and x' defined by the convexity structure is identical with that defined by the algebraic structure $\mathcal{L}(Y) \times \mathcal{L}(L)$.

To do this we write $x = (y, \ell)$ and $x' = (y', \ell')$. If $\dim\{\theta, e_{n+1}, y, y'\} = 1$ then $x, x' \in L$ and our afirmation is trivial.

If $\dim\{\theta, e_{n+1}, y, y'\} = 2$ then $(e_2, \ldots, e_n) \notin \{\theta, e_{n+1}, y, y'\}$. In fact if we assume contrary then $\{\theta, e_2, \ldots, e_n, e_{n+1}\} \subset \subset \{\theta, e_{n+1}, y, y'\}$ and therefore $n \leq 2$ which is a contradiction. So we may suppose in this case $e_2 \notin \{\theta, e_{n+1}, y, y'\}$.

Define

$$Y_2 = \begin{cases} \{\theta, e_{n+1}, y, y'\} & \text{, if } \dim\{\theta, e_{n+1}, y, y'\} = 3 \\ \{\theta, e_2, e_{n+1}, y, y'\}, & \text{if } \dim\{\theta, e_{n+1}, y, y'\} = 2 \text{ and} \\ & e_2 \notin \{\theta, e_{n+1}, y, y'\} . \end{cases}$$

Since $\dim Y_2 = 3$ there exists a strong real linearization $\mathcal{L}(Y_2)$ with the origin θ for Y_2. Applying two times the dimension formula ([3], theorem 9) we obtain :

(17) $$n+1 \geq \dim\{Y_2, Y_1\} = \dim Y_2 + \dim Y_1 - \dim Y_2 \cap Y_1 = 3 + n - \dim Y_2 \cap Y_1$$

and

(18) $$n+1 = \dim\{Y_2, Y\} = \dim Y_2 + \dim Y - \dim Y_2 \cap Y = 3 + n - \dim Y_2 \cap Y .$$

From (17) and (18) we have

(19) $$\dim Y_2 \cap Y_1 \geq 2 ,$$

(20) $$\dim Y_2 \cap Y = 2 .$$

(19) and (20) permit to apply to the spaces $Y_2 \cap Y_1$ and $Y_2 \cap Y$ the uniquence theorem (B) on the linearization. Thus

(21) $$\mathcal{L}(Y_2)\big|_L = \mathcal{L}(Y_1)\big|_L = \mathcal{L}(L)$$

(22) $$\mathcal{L}(Y_2)\big|_{Y_2\cap Y} = \mathcal{L}(Y)\big|_{Y_2\cap Y}$$

and since $Y_2 = (Y_2\cap Y)\times L$, $\mathcal{L}(Y_2) = \mathcal{L}(Y_2)\big|_{Y_2\cap Y}\times \mathcal{L}(Y_2)\big|_L$ and $(\mathcal{L}(Y)\times\mathcal{L}(L))\big|_{(Y_2\cap Y)\times L} = \mathcal{L}(Y)\big|_{Y_2\cap Y}\times\mathcal{L}(L)$, from (21), (22) we have

(23) $$\mathcal{L}(Y_2) = (\mathcal{L}(Y)\times\mathcal{L}(L))\big|_{(Y_2\cap Y)\times L} .$$

Since $x,x'\in Y_2$ it follows that all the open segment $x\cdot x'$ is contained in Y_2 and since $\mathcal{L}(Y_2)$ is a strong real linearization of Y_2, this segment coincides with the algebraic open segment defined by $\mathcal{L}(Y_2)$ which by (23) coincides with the algebraic open segment defined by $\mathcal{L}(Y)\times\mathcal{L}(L)$.

Therefore $\mathcal{L}(Y)\times\mathcal{L}(L)$ is a strong linearization of X.

Case 2. $\dim X = \infty$. Consider B a basis in X; θ, $e\in B$, $\theta\neq e$. Then $Y = \{B\setminus e\}$ is a hyperplane in X, $L = \{\theta, e\}$ is a line and $Y\cap L = \theta$. As in the case 1 it is possible the representation $X = Y\times L$.

The convexity structures induced on Y and L have the following properties :

(24) $y_1\cdot y_2 = \pi_1\ (y_1,l_1)\cdot(y_2,l_2)$ $\quad(\forall) l_1,l_2\in L$

(25) $y_1/y_2 = \pi_1\ (y_1,l_1)/(y_2,l_2)$ $\quad(\forall) l_1,l_2\in L$

(26) $l_1\cdot l_2 = \pi_2\ (y_1,l_1)\cdot(y_2,l_2)$ $\quad(\forall)\ y_1,y_2\in Y$

(27) $l_1/l_2 = \pi_2\ (y_1,l_1)/(y_2,l_2)$ $\quad(\forall)\ y_1,y_2\in Y$

where for $A\subset Y\times L$ we define $\pi_1 A = (y\in Y\colon (\exists) l\in L$ such that $(y,l)\in A)$ and $\pi_2 A = (l\in L\colon (\exists)\ y\in Y$ such that $(y,l)\in A)$.

In the following we identify the line L with the real numbers set in the following way : we choose $e_1,e_2\in B\setminus(\theta,e)$ with $e_1\neq e_2$ and we denote $X_1 = \{\theta, e_1,e_2,e\}$. From $\dim X_1 = 3$ it follows that X_1 admits a strong real linearization with origin in θ : $\mathcal{L}(X_1)$. In the next we consider as real coordinatization of L the coordinatization induced by $\mathcal{L}(X_1)$; so $\mathcal{L}(L) = \mathcal{L}(X_1)\big|_L$.

Using this coordinatization of L we can define the set Y^* of all functions $f:Y \longrightarrow \mathbb{R} \equiv L$ with $f(\theta)=0\equiv\theta$, for which graph $(f) = ((y,f(y)):y \in Y) \subset Y \times L = X$ is a linear set of X. Our next step is to prove that Y^* is a linearization family for Y.

First, observe that for $f \in Y^*$ and $y_1, y_2 \in Y$ we have:

(28) $$f(y_1 \cdot y_2) = f(y_1) \cdot f(y_2) \ ,$$

(29) $$f(y_1/y_2) = f(y_1)/f(y_2) \ .$$

To verify this let $y \in y_1 \cdot y_2$. By (24) there exists $\ell \in L$ such that $(y,\ell) \in (y_1,f(y_1)) \cdot (y_2,f(y_2))$. Since graph (f) is linear then $(y_1,f(y_1)) \cdot (y_2,f(y_2)) \subset$ graph(f) and so $(y,\ell) \in$ graph(f). Therefore $\ell = f(y)$. Using (26) we obtain $f(y) \in f(y_1) \cdot f(y_2)$ which proves that $f(y_1 \cdot y_2) \subset f(y_1) \cdot f(y_2)$.

Fot the converse inclusion let $\ell \in f(y_1) \cdot f(y_2)$. Then by(26) there exists $y \in Y$ such that $(y,\ell) \in (y_1,f(y_1)) \cdot (y_2,f(y_2)) \subset$ graph(f). Thus $\ell = f(y)$ and by (24) $y \in y_1 \cdot y_2$. So $f(y_1) \cdot f(y_2) \subset f(y_1 \cdot y_2)$ and (28) is proved.

To prove (29) we use the same reasonement with "/" instead of the operation "·" .

According to (28) we see that Y^* satisfies (8).

For (9) we have to prove

(30) $$(y_1, y_2 \in Y,\ y_1 \neq y_2) \Longrightarrow (\exists f)(f \in Y^*,\ f(y_1) \neq f(y_2))$$

For, let Y_1 be a hyperplane in Y such that $\theta \in Y_1$, $Y_1 \cap \{y_1,y_2\} \neq \emptyset$ and $\{y_1,y_2\} \not\subset Y_1$. We may assume $y_2 \notin Y_1$. Let $F = \{Y_1,(y_2,1)\} = Y_1 \cup (y_2,1) \cdot Y_1/Y_1 \cup Y_1/(y_2,1)$. We have

$$\{F,(y_2,0)\} = \{\{Y_1,(y_2,0)\},\ (y_2,1)\} = \{Y,(y_2,1)\} = X \ .$$

In addition $(y_2,0) \notin F$ because otherwise either $(y_2,0) \in (y_2,1) \cdot Y_1/Y_1$ or $(y_2,0) \in Y_1/(y_2,1)$. In the first case it would follow that $(y_2,0) \cdot Y_1 \cap (y_2,1) \cdot Y_1 \neq \emptyset$ and in the second case that $(y_2,0) \cdot (y_2,1) \cap Y_1 \neq \emptyset$. According to (26) in both cases we w obtain $0 \in 1 \cdot 0$ which is absurd.

Thus by [2] lemma(16),pp.325, F is a hyperplane of X with $\theta \in F$.

If we shall prove that for any $y \in Y$

(31) $$\text{card } F \cap L_y = 1 \ ,$$

where L_y is the parallel through y to L, then the existence of $f \in Y^*$ such that graph(f) = F will be proved.

First observe that $L \not\subset F$, because otherwise, from the fact that $L_{y_2} \| L \subset F$ and $L_{y_2} \cap F \neq \emptyset$ it would follow $L_{y_2} \subset F$. Thus $(y_2, 0) \in F$, but this is impossible by what was proved above.

Now, if for $y \in Y$ the line L_y was parallel to F it would follow that L is also parallel to F and by $L \cap F \neq \emptyset$, that $L \subset F$, in contradiction. Therefore $F \cap L_y \neq \emptyset$.

Finally, observe that if card $F \cap L_y > 1$ then $L_y \subset F$ and since $L_y \| L$ and $L \cap F \neq \emptyset$, it would have $L \subset F$, which is again a contradiction. Thus (31) holds and for each $y \in Y$ we may define $f(y) = \pi_2\, F \cap L_y$.

We have only to show that $f(y_1) \neq f(y_2)$. Denote $y = Y_1 \cap \{y_1, y_2\}$, then $f(y)=0$. If $y=y_1$ then $f(y_1)=f(y)=0\neq 1=f(y_2)$ and so (30) is proved. If now $y \neq y_1$ then since $y_2 \notin Y_1$ we have $y \neq y_2$. The following cases are possible: $y \in y_1 \cdot y_2$, $y_1 \in y \cdot y_2$, $y_2 \in y \cdot y_1$. In each of them by (28): $0 \in f(y_1) \cdot 1$, $f(y_1) \in 0 \cdot 1$, $1 \in 0 \cdot f(y_1)$, respectivelly. Thus in all cases $f(y_1) \neq f(y_2)$ which completes the proof for (30).

To verify (10) let be the line $\{y_1, y_2\} = y_1/y_2 \cup y_1 \cup y_1 \cdot y_2 \cup y_2 \cup y_2/y_1$, where $y_1 \neq y_2$. If $f(y_1) = f(y_2) = \ell$, then the fact that $f(y) = \ell$ for every $y \in \{y_1, y_2\}$ is a trivial concequence of (28), (29), (13) ; thus in this case f restricted to $\{y_1, y_2\}$ is a constant map.

In the case $f(y_1) \neq f(y_2)$ the injectivity of this restriction is trivial. For the surjectivity let $r \in \mathbb{R} = \{f(y_1), f(y_2)\}$, $f(y_1) \neq r \neq f(y_2)$. Then $r \in f(y_1)/f(y_2) = f(y_1/y_2)$ or $r \in f(y_1) \cdot f(y_2) = f(y_1 \cdot y_2)$ or $r \in f(y_2)/f(y_1) = f(y_2/y_1)$ and consequently $r \in f(\{y_1, y_2\})$. Thus (10) is verified.

To prove (11) consider $y_1, y_2 \in Y$, $y_1 \neq y_2$, the line $D = \{y_1, y_2\}$ and the finite dimensional subspace $X_2 = \{X_1, D\}$ of X. X_2 has an unique strong real linearization with origin θ: $\mathcal{L}(X_2)$. Since X_1 admits an unique linearization with the origin θ we may write $\mathcal{L}(X_2)|_L = (\mathcal{L}(X_2)|_{X_1})|_L = \mathcal{L}(X_1)|_L = \mathcal{L}(L)$. This means that the coordinatization of L induced by $\mathcal{L}(X_2)$ is identical with that induced by $\mathcal{L}(X_1)$.

Let $f, g \in Y^*$. The restrictions to $\pi_1 X_2$ of f and g are linear functions since their graphs in $\pi_1 X_2 \times L$ are linear sets.

If f separates y_1 and y_2 then the following system

$$\begin{cases} \lambda f(y_1) + \mu = g(y_1) \\ \lambda f(y_2) + \mu = g(y_2) \end{cases}$$

has an unique solution (λ, μ).

If $y \in \{y_1, y_2\}$ then there is $\alpha \in \mathbb{R}$ such that $y = \alpha y_1 + (1-\alpha) y_2$ (in $\mathcal{L}(X_2)$). But then

$$g(y) = \alpha g(y_1) + (1-\alpha) g(y_2) = \alpha(\lambda f(y_1) + \mu) + (1-\alpha)(\lambda f(y_2) + \mu) =$$
$$= \lambda(\alpha f(y_1) + (1-\alpha) f(y_2)) + \mu = \lambda f(y) + \mu$$ and the proof of (11) is completed.

To finish the proof of our theorem we must only show that the real linear structure $\mathcal{L}(Y) \times \mathcal{L}(L)$ is a strong real linearization of $Y \times L = X$.

For, consider $x_1, x_2 \in X$, $x_1 \neq x_2$, $X_3 = \{X_1, x_1, x_2\}$, $X_4 = X_3 \cap Y$. Since $3 \leq \dim X_3 < \infty$ there exists a strong real linearization with the origin Θ of X_3 and since $\dim X_3 \geq \dim X_4 \geq 2$ we have $\mathcal{L}(X_3)|_{X_4} = \mathcal{L}(Y)|_{X_4}$ and $\mathcal{L}(X_3)|_{X_1} = \mathcal{L}(X_1)$. Therefore

$$\mathcal{L}(X_3) = \mathcal{L}(X_3)|_{X_4} \times \mathcal{L}(X_3)|_L = \mathcal{L}(Y)|_{X_4} \times (\mathcal{L}(X_3)|_{X_1})|_L =$$
$$= \mathcal{L}(Y)|_{X_4} \times \mathcal{L}(X_1)|_L = \mathcal{L}(Y)|_{X_4} \times \mathcal{L}(L) = (\mathcal{L}(Y) \times \mathcal{L}(L))|_{X_4 \times L}$$

Thus

$$\mathcal{L}(X_3) = (\mathcal{L}(Y) \times \mathcal{L}(L))|_{X_4 \times L}. \tag{32}$$

Since $x_1 \cdot x_2 \subset X_3$ and $\mathcal{L}(X_3)$ is a strong real linearization of X_3 it follows that $x_1 \cdot x_2$ coincides with the algebraic segment induced by $\mathcal{L}(X_3)$ and next by (32) it coincides with the algebraic segment induced by $\mathcal{L}(Y) \times \mathcal{L}(L)$. Therefore $\mathcal{L}(Y) \times \mathcal{L}(L)$ is a strong real linearization of X.

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