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QUASI-CONVEXITY IN LINEAR SPACES

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1. In the paper [3] we have defined a quasi-convexity property in abstract (nonlinear) spaces. For, we have considered a set X, three subsets $S_o \subset S_1 \subset S_2 \subset X$, two partitions of S_1 and S_2 :

(1) $S_1 = S_1^- \bigcup \tilde{S}_o \bigcup S_1^+$, where $S_o \subset \tilde{S}_o$,

(2) $S_2 = S_2^- \bigcup \tilde{S}_1 \bigcup S_2^+$, where $S_1 \subset \tilde{S}_1$

and two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ of interpolating operators relative to S_1, respectively to S_2, that is :

$$U_1 \in \mathcal{U}_1 \Rightarrow U_1 : X \to S_1 \text{ and } U_1x = x \text{ for any } x \in S_1 ;$$

$$U_2 \in \mathcal{U}_2 \Rightarrow U_2 : X \to S_2 \text{ and } U_2x = x \text{ for any } x \in S_2 .$$

In the set of operators $U : X \to X$ one defin es [1] the relation :

(3) $U' \lhd U \iff UU' = U'U = U'$.

DEFINITION 1 [3] . <u>We say that the triplet (U_1', U_1, U_1'') is a decomposition of the operator $U_2 \in \mathcal{U}_2$ relative to (S_o, S_1), if :</u>

1° <u>$U_1', U_1, U_1'' \in \mathcal{U}_1$, $U_1', U_1, U_1'' \lhd U_2$</u> ;

2° <u>For each $y \in S_2^+$ (S_2^-) such that $U_1y \in S_o$, we have $U_1'y \in S_1^-$ (S_1^+) and $U_1''y \in S_1^+$ (S_1^-)</u> .

DEFINITION 2 [3] . <u>The element $x \in X$ is said to be (S_o, S_1, S_2) quasi-convex if for each $U_2 \in \mathcal{U}_2$ for which there is a decomposition</u>

<u>(U_1', U_1, U_1'') relative to (S_0, S_1) such that $U_1 x \in S_0$, the following relation is satisfied :</u>

(4) $$U_2\, x \in S_2^+ .$$

In [3] we have proved that (4) implies $U_1' x \in S_1^-$ and $U_1'' x \in S_1^+$ and that if $\widetilde{S}_1 = S_1$ then the converse implication is also true.

2. In this paper we deal with the linear case. Let X be a real linear space, $S_0 \subsetneqq S_1 \subsetneqq S_2$ three linear subspaces of X, S_1 a maximal proper subspace of S_2 and $\widetilde{S}_0$ a maximal proper subspace of S_1, with $S_0 \subset \widetilde{S}_0$.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the sets of linear interpolating operators relative to S_1 and S_2, respectively. If we fix $y_1 \in S_1 \setminus \widetilde{S}_0$ and $y_2 \in S_2 \setminus S_1$, then we can associate [1] to each operator $U_1 \in \mathcal{U}_1$ the divided difference functional $[U_1; \cdot] : X \longrightarrow \mathbb{R}$ which satisfies : $[U_1\, ;\, U_1 x] = [U_1; x]$ for all $x \in X$, $[U_1; x] = 0$ for any $x \in \widetilde{S}_0$ and $[U_1\, ;\, y_1] = 1$ and to each operator $U_2 \in \mathcal{U}_2$, the divided difference $[U_2\, ;\, \cdot] : X \longrightarrow \mathbb{R}$ satisfying $[U_2\, ;\, U_2 x] = [U_2\, ;\, x]$ for all $x \in X$, $[U_2\, ;\, x] = 0$ for any $x \in S_1$ and $[U_2\, ;\, y_2] = 1$.

In [1] it is shown that for all $U_2 \in \mathcal{U}_2$, $U_1, U_1' \in \mathcal{U}_1$ such that $U_1, U_1' \lhd U_2$ and $[U_1\, ;\, \cdot] \neq [U_1'\, ;\, \cdot]$, the following recurrence formula is satisfied :

(5) $$[U_2\, ;\, x] = \frac{[U_1;x] - [U_1';x]}{[U_1;y_1] - [U_1';y_1]} \qquad (x \in X) .$$

3. Under the assumptions from section 2, define :

$$S_1^+ = \{x \in S_1 \mid [U_1\, ;\, x] > 0,\ \forall\, U_1 \in \mathcal{U}_1\}, \qquad S_1^- = -\, S_1^+ ,$$

$$S_2^+ = \{x \in S_2 \mid [U_2\, ;\, x] > 0,\ \forall\, U_2 \in \mathcal{U}_2\}, \qquad S_2^- = -\, S_2^+$$

and consider the following partitions:

$$S_1 = S_1^- \cup \widetilde{S}_0 \cup S_1^+ , \qquad S_2 = S_2^- \cup S_1 \cup S_1^+ .$$

LEMMA 1. <u>If $U_2 \in \mathcal{U}_2$, then the following propositions are true</u> :

1° <u>The triplet (U_1', U_1, U_1''), where $U_1', U_1, U_1'' \in \mathcal{U}_1$ and $U_1', U_1, U_1'' \lhd U_2$, is a decomposition of U_2 relative to (S_o, S_1), if and only if there exists $y \in S_2^+$ satisfying $U_1 y \in S_o$, $U_1' y \in S_1^-$ and $U_1'' y \in S_1^+$.</u>

2° <u>If $S_o' \subset \tilde{S}_o$, then each decomposition of U_2 relative to (S_o, S_1), is equally a decomposition of U_2 relative to (S_o', S_1).</u>

3° <u>If (U_1', U_1, U_1'') is a decomposition of U_2 relative to (S_o, S_1), then for each $y \in S_2^+$ we have</u>

(6) $$[U_1' ; y] < [U_1 ; y] < [U_1'' ; y].$$

Proof. 1° Assume that there exists $y \in S_2^+$ such that $U_1 y \in S_o$, $U_1' y \in S_1^-$ and $U_1'' y \in S_1^+$ and consider $y' \in S_2^+$ such that $U_1 y' \in S_o$. By (5) we have:

$$0 < [U_2; y] = \frac{[U_1'; y] - [U_1; y]}{[U_1'; y_2] - [U_1; y_2]}, \quad 0 < [U_2; y'] = \frac{[U_1'; y'] - [U_1; y']}{[U_1'; y_2] - [U_1; y_2]}$$

and since $[U_1; y] = [U_1; y'] = 0$, we deduce that $[U_1'; y']$ has the same sign as $[U_1'; y]$. Similarly, $[U_1''; y']$ and $[U_1''; y]$ have the same sign.

To prove 2° we have to show that the decompositions relative to (S_o, S_1) and those relative to $(\tilde{S}_o, S_1)$ are the same.

Obviously, each decomposition relative to $(\tilde{S}_o, S_1)$ is also a decomposition relative to (S_o, S_1).

Let (U_1', U_1, U_1'') be a decomposition of U_2 relative to (S_o, S_1) and $y \in S_2^+$ such that $U_1 y \in \tilde{S}_o$. If we put $y' = y - U_1 y$, then $y' \in S_2^+$ and $U_1 y' = 0 \in S_o$. Consequently, $U_1' y' \in S_1^-$ and $U_1'' y' \in S_1^+$, that is $U_1' y - U_1 y \in S_1^-$ and $U_1'' y - U_1 y \in S_1^+$. Thus $U_1' y \in S_1^-$ and $U_1'' y \in S_1^+$. Finally, we take into account 1°.

3° For $y \in S_2^+$ define $y' = y - U_1 y \in S_2^+$. Since $U_1 y' = 0 \in S_o$

it follows $U_1'y' \in S_1^-$ and $U_1''y' \in S_1^+$. Therefore $[U_1';U_1'y'] < 0$ and $[U_1'';U_1''y'] > 0$, that is $[U_1';y] - [U_1;y] < 0$ and $[U_1'';y] - [U_1;y] > 0$.

LEMMA 2. Let $U_2 \in \mathcal{U}_2$, $U_1', U_1, U_1'' \in \mathcal{U}_1$, $U_1', U_1, U_1'' \lhd U_2$. (U_1', U_1, U_1'') is a decomposition of U_2 relative to (S_0, S_1) if and only if there exists $y \in S_2^+$ such that (6) be satisfied.

Proof. The necessity of the condition follows by lemma 1.3°. To prove that the condition is sufficient, assume that there exists $y \in S_2^+$ satisfying (6). Then, $y' = y - U_1 y \in S_2^+$, $[U_1;y'] = 0$, $[U_1';y'] < 0$ and $[U_1'';y'] > 0$. Therefore, $U_1'y' \in S_1^-$ and $U_1''y' \in S_1^+$ and we can apply lemma 1.

THEOREM 1. Let $x \in X$. The following propositions are equivalent:

1° x is (S_0, S_1, S_2) - quasi - convex;

2° for all $U_2 \in \mathcal{U}_2$, $U_1', U_1, U_1'' \in \mathcal{U}_1$ with $U_1', U_1, U_1'' \lhd U_2$, $[U_1;x] = 0$, for which there exists $y \in S_2^+$ satisfying (6), the following relations hold :

$$[U_1' ; x] < 0 < [U_1'' ; x].$$

The proof is immediate by lemma 2.

DEFINITION 3. The element $x \in X$ is called strong (S_0, S_1, S_2) - quasi-convex if for each decomposition (U_1', U_1, U_1'') relative to (S_0, S_1) of some operator from $\mathcal{U}_2$, the following inequality is satisfied :

$$0 < \max\{-[U_1' ; x], [U_1'' ; x]\}.$$

THEOREM 2. Any strong (S_0, S_1, S_2) - quasi-convex element is (S_0, S_1, S_2) - quasi-convex.

Proof. Let $x \in X$ be a strong (S_0, S_1, S_2)-quasi-convex element, $U_2 \in \mathcal{U}_2$ and (U_1', U_1, U_1'') a decomposition of U_2 relative to (S_0, S_1), which satisfies $U_1 x \in S_0$ and (8). If $[U_1'' ; x] = \max\{-[U_1';x], [U_1'';x]\}$, then $[U_1'';x] > 0 = [U_1; x]$.

By (5) : $\left[U_2 ; x\right] = \dfrac{\left[U_1'' ; x\right] - \left[U_1 ; x\right]}{\left[U_1'' ; y_2\right] - \left[U_1 ; y_2\right]}$

and since $y_2 \in S_2^+$, the denominator of the fraction is positive. Therefore, $[U_2;x] > 0$. The same inequality holds even if $\max\left\{-\left[U_1' ; x\right], \left[U_1'' ; x\right]\right\} = -\left[U_1' ; x\right]$.

In a following paper we will deal with the conditions so that the converse of theorem 2 be also true.

EXEMPLE. Let X be the set of all real functions f defined on the interval $\mathcal{J} \subset R$, $S_0 = \mathcal{P}_0$, $S_1 = \mathcal{P}_n$, $S_2 = \mathcal{P}_{n+1}$, $S_1^- = \mathcal{P}_n^-$, $\tilde{S}_0 = \mathcal{P}_{n-1}$, $S_1^+ = \mathcal{P}_n^+$, $S_2^- = \mathcal{P}_{n+1}^-$, $S_2^+ = \mathcal{P}_{n+1}^+$, where $\mathcal{P}_k (k \geq 0)$ denotes the set of the polynomials by degree $\leq k$ and $\mathcal{P}_k^+$ ($\mathcal{P}_k^-$) is the subset of all the elements of $\mathcal{P}_k$ having positive (negative) dominant coefficient.

Denote by $\mathcal{U}_1$ the set of all Lagrange interpolating operators $L(\mathcal{P}_n; x_1, x_2, \ldots, x_{n+1}; \cdot)$ and by $\mathcal{U}_2$ the set of all Lagrange interpolating operators $L(\mathcal{P}_{n+1}; x_1, x_2, \ldots, x_{n+2}; \cdot)$, where $x_1, x_2, \ldots, x_{n+2}$ are distinct points in $\mathcal{J}$. The divided differences associated with these operators are those ordinary, on the points $x_1, x_2, \ldots, x_{n+1}$, respectively on $x_1, x_2, \ldots, x_{n+2}$.

Let $U_2 = L(\mathcal{P}_{n+1}; x_1, x_2, \ldots, x_{n+2}; \cdot) \in \mathcal{U}_2$, where $x_1 < x_2 < \ldots < x_{n+2}$.

LEMMA 3. The triplet (U_1', U_1, U_1'') is a decomposition of U_2 relative to $(\mathcal{P}_0, \mathcal{P}_n)$, if and only if

$$U_1' = L(\mathcal{P}_n; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}; \cdot)$$
$$U_1 = L(\mathcal{P}_n; x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+2}; \cdot)$$
$$U_1'' = L(\mathcal{P}_n; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+2}; \cdot)$$

where $1 \leq k < j < i \leq n+2$.

Proof. Denote by $D_i(f)$ the divided difference $[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}; f]$, $i = 1, 2, \ldots, n+2$. One has

$$(x_{n+2} - x_1) D_k(f) = (x_k - x_1) D_{n+2}(f) + (x_{n+2} - x_k) D_1(f),$$

$k = 1, 2, \ldots, n+2$. From hence, we infer that

$$(x_{n+2} - x_1)(D_i(f) - D_j(f)) = (x_i - x_j)(D_{n+2}(f) - D_1(f)).$$

Now, if f is a convex function of order n on $\mathcal{J}$, then $D_{n+2}(f) < D_1(f)$ and consequently $D_i(f) < D_j(f)$ for any i and j such that $1 \leq j < i \leq n+2$. Thus,

$$D_{n+2}(f) < D_{n+1}(f) < \ldots < D_2(f) < D_1(f)$$

and the proof is complete if we take into account lemma 2 and lemma 1 [3].

THEOREM 3. The function $f : \mathcal{J} \to \mathbb{R}$ is strong$(\mathcal{P}_0, \mathcal{P}_n, \mathcal{P}_{n+1})$-quasi-convex if and only if for each system of points $x_1 < x_2 < \ldots < x_{n+2}$ from $\mathcal{J}$ and for every i and k such that $1 \leq k < i \leq n+2$, $i-k \geq 2$, the following inequality is satisfied:

$$0 < \max\{-[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+2};f],\ [x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+2};f]\}. \tag{9}$$

The proof is immediate by lemma 3.

Let us remark that if the function f is strong $(\mathcal{P}_0, \mathcal{P}_n, \mathcal{P}_{n+1})$-quasi-convex, then it satisfies

$$0 < \max\{-[x_1,x_2,\ldots,x_{n+1};f],\ [x_2,x_3,\ldots,x_{n+2};f]\}$$

and the equivalent inequality

$$[x_2,x_3,\ldots,x_{n+1};f] < \max\{[x_1,x_2,\ldots,x_n;f],\ [x_3,x_4,\ldots,x_{n+2};f]\}.$$

This means that each strong $(\mathcal{P}_0, \mathcal{P}_n, \mathcal{P}_{n+1})$ - quasi-convex function is „strictly quasi-convex of order n-1" in the sense of [2].

REFERENCES

[1] Ivan, M., Procedee de interpolare şi aplicaţii ale lor (Dissertation), Cluj-Napoca, 1982.

[2] Popoviciu, E. Sur une allure de quasi-convexité d'ordre supérieure, L'analyse num. et la th.de l'approx., 11, 129-137 (1982).

[3] Precup, R., Sur une notion de quasi-convexité dans des espaces abstraits, Itinerant Seminar on Functional Equations, Approx.and Convexity, Cluj-Napoca, 1984, 143 - 150.