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MONOTONICITY PROPERTIES OF THE BEST APPROXIMATION OPERATORS

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Let X and Y be two real linear spaces, $\langle \cdot , \cdot \rangle : X \times Y \to \mathbb{R}$ — a bilinear functional, $K \subset X$ — a convex cone, K^* — the polar cone of K, that is $K^* = \{ y \in X ; \langle x,y \rangle \geq 0$ for all $x \in K \}$ and let A be an operator from X into Y.

The operator A is said to be monotone if it satisfies the following condition

(1) $\langle x - x', Ax - Ax' \rangle \geq 0$ for all $x, x' \in X$

and is said to be (o)-monotone if

(2) $Ax - Ax' \in K^*$ for all $x, x' \in X$ such that $x - x' \in K$.

In our previous paper [2] we have defined a larger class of operators as follows

DEFINITION [2]. The operator $A : X \to Y$ is K-monotone if

(3) $\langle x - x', Ax - Ax' \rangle \geq 0$ for all $x, x' \in X$ such that $x - x' \in K$.

The monotonicity, (o)-monotonicity and K-monotonicity properties can be extended to multivalued mappings.

Obviously, any monotone or (o)-monotone mapping is a K-monotone one. Therefore, the study of the K-monotone mappings is important for a synthesis of some results in monotone and in (o)-monotone mappings.

In this paper we deal with the monotonicity properties of some best approximation operators.

In what follows X is a real Hilbert space, $Y = X$ and the bilinear functional $\langle \cdot , \cdot \rangle$ is the scalar product on X. Denote by $\| \cdot \|$ the norm in X induced by the scalar product, by $\| \cdot \|'$ an arbitrary norm in X, equivalent with $\| \cdot \|$ and by C a nonvoid closed convex subset of X.

The mapping $B = B(\| \cdot \|', C)$, $B : X \to 2^X$ assigning to each $x \in X$ the subset $\{ y \in C ; \|x - y\|' = \inf_{z \in C} \|x - z\|' \}$ is called the best approximation mapping by elements of C and with respect to the norm $\| \cdot \|'$.

Since the space $(X, \| \cdot \|')$ is reflexive, we have $Bx \neq \emptyset$ for all $x \in X$. Moreover, if the norm $\| \cdot \|'$ or the subset C is strictly convex, then for each $x \in X$ the subset Bx contains an unique element denoted by x^*; in this case the mapping B is an operator from X into X.

It is well known that the operator $B(\| \cdot \|, C)$ is monotone for every nonvoid closed convex subset C.

For other norms $\| \cdot \|'$ the operators $B(\| \cdot \|', C)$ can be not monotone, but under some additional geometrical conditions imposed to the norms and to the subsets C, they can be (o)-monotone or only K-monotone with respect to certain cones K.

For an example of best approximation operator which is not monotone but is (o)-monotone, let us consider in $X = \mathbb{R}^2$ the norm $\|(x,y)\|_\infty = \max \{ |x| , |y| \}$ and a closed strictly convex subset $C \subset \{ (x,y) ; \|(x,y)\|_\infty \leq 1 \}$ such that the points $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$ belong to its boundary ∂C.

Let $(x,y) \notin C$. Then $B(x,y) = (-1,0)$ if $y + x + 1 \leq 0$ and $y - x - 1 \geq 0$; $B(x,y) = (1,0)$ if $y - x + 1 \leq 0$ and $y + x - 1 \geq 0$; $B(x,y) = (0,-1)$ if $y - x + 1 \leq 0$ and $y + x + 1 \leq 0$; $B(x,y) = (0,1)$ if $y + x - 1 \geq 0$ and $y - x - 1 \geq 0$. Also, if $y - x - 1 < 0$, $y - x + 1 > 0$, then $B(x,y) = (x^*, y^*)$ where $y^* - y = x^* - x$, $x \cdot x^* > 0$ and $(x^*, y^*) \in \partial C$; while if $y + x + 1 > 0$ and $y + x - 1 < 0$, then $B(x,y) = (x^*, y^*)$ where $y^* - y = x - x^*$, $x \cdot x^* > 0$ and $(x^*, y^*) \in \partial C$.

To prove that the operators B and $(-B)$ are not monotone, let us consider the point $z = (x , x) \in \partial C$ with $x > 0$. Then, by using the proprieties of C, we see that $1/2 < x < 1$. Also, let $x' \geq 1$ and let $z' = (x' , x' - 1)$. One has $Bz = z$ and $Bz' = (1,0)$. Consequently, $\langle z' - z, Bz' - Bz \rangle = 2x^2 - (2x - 1)x'$ and since $2x - 1 > 0$, we have that $\langle z' - z, Bz' - Bz \rangle < 0$ (>0) for $x' > 2x^2/(2x - 1)$ (respectively for $x' < 2x^2/(2x - 1)$). Thus , the operators B and $(-B)$ are not monotone.

Under an additional condition on the set C , the operator $B(\|\cdot\|_\infty, C)$ will be (o)-monotone.

In order to formulate this condition, for each number $\alpha > 0$ let us consider the convex cone (with nonvoid interior)

$$K_\alpha = \{(x,y) \in \mathbb{R}^2 \; ; \; |y| \leq \alpha x\}.$$

Now we can state :

<u>If the subset $C \subset \mathbb{R}^2$ satisfies the above conditions and in addition there is $\alpha > 1$ such that $C \subset (z_0 - K_\alpha) \cap (-z_0 + K_\alpha)$, where $z_0 = (1,0)$, then the operator $B(\|\cdot\|_\infty, C)$ is (o)-monotone with respect to the cones $K_{1/\alpha}$ and $(K_{1/\alpha})^* = K_\alpha$.</u>

Returning at the case of an arbitrary Hilbert space X , we will give an example of K-monotone operator which is not monotone and (essentially) nor (o)-monotone.

For this, let us consider the strictly convex set $C = \{x \in X \; ; \|x\| \leq 1\}$, a fixed element $u \in X$, $\|u\| = 1$ and the norm $\|\cdot\|_u$ associated to u as follows :

$$(4) \quad \|x\|_u = \|x\| \quad ; \text{ if } |\langle x,u\rangle| \geq \|x\|/\sqrt{2}$$
$$= \|x_u\|\sqrt{2}, \text{ if } |\langle x,u\rangle| < \|x\|/\sqrt{2} ,$$

where $\tilde{x}_u = x - \langle u,x \rangle u$.

If we denote by A the operator $B(\|\cdot\|_u, C)$ and we consider the following convex cone

(5) $$K_u = \{x \in X ; \langle x, u \rangle \geq \|x\|/\sqrt{2}\},$$

then we can state

THEOREM. 1^o. The operators A and $(-A)$ are not monotone;

2^o. For each convex cone $K \subset X$ with $\dim K \geq 2$, the operators A and $(-A)$ are not (o)-monotone ;

3^o. The operator A is K_u - monotone .

Details will appear in [4] .

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