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NONLINEAR BOUNDARY VALUE PROBLEMS FOR INFINITE SYSTEMS OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

by

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1. INTRODUCTION.

Let X be a real Banach space with norm $|\cdot|$ and let I be the interval $[0,1]$. Let

$$f : I \times X \times X \times C(I;X) \longrightarrow X \ ,$$

$$W_1 u = a\, u(0) - b\, u'(0) \ ,$$

and $$W_2 u = c\, u(1) + d\, u'(1) \qquad (u \in C(I;X)) \ ,$$

where a,b,c,d are nonnegative real numbers with $a + b > 0$ and $c + d > 0$. Consider the mappings

$$V_1 u = \varphi(u(0), u'(0), u(1), u'(1)) \ ,$$

$$V_2 u = \psi(u(0), u'(0), u(1), u'(1)) \quad (u \in C^1(I;X)),$$

where $\varphi, \psi : X^4 \longrightarrow X$.

This paper is concerned with the existence of solutions of the boundary value problem

$$(1.1) \qquad u'' = f(t,u,u';u) \qquad (t \in I) \ ,$$

$$(1.2) \qquad W_i u = V_i u \qquad (i = 1,2).$$

By a solution of (1.1) we mean a function $u \in C^2(I;X)$ satisfying $u''(t) = f(t,u(t),u'(t);u)$ for all $t \in I$.
We shall require that f, φ and ψ be α- Lipschitz, α being the Kuratowski's measure of noncompactness. There are more reasons for the investigation of equation (1.1). One of them is the interest in the existence of solutions to some boundary value problems for systems of second-order differential equations with deviating arguments.

The technique that we shall use is based on the Leray - Schauder's alternative for condensing mappings and does not use the topological degree. In addition, we shall use the "a priori bounds" method.

Our paper has as point of starting, on the one hand the monograph [7] of A.Granas, R.Guenther and J.Lee, which in case X is finite-dimensional is concerned with the system

$$u'' = f(t,u,u') \tag{1.3}$$

with boundary conditions (1.2) assuming the continuity of f, φ and ψ, and on the other hand the paper [18] of K.Schmitt and R.Thompson, which deals with the existence of solutions to the infinite system (1.1) satisfying
$u(0) = x_0$ and $u(1) = x_1$ $(x_0, x_1 \in X)$ or the boundary conditions $W_i u = 0$ $(i = 1,2)$, assuming the complete continuity of f.

The Leray-Schauder's alternative on completely continuous mappings (see [3]) has been extended for condensing mappings by M.Martelli and A.Vignoli [12]. Their proof uses the topological degree in the sence of R.D.Nussbaum [14] (see also [17,pp.113 - 121]). In Section 2 we shall give a proof to this extention without using the topological degree, but only the elementary notion of essential mapping (see [6]).

In Section 3 we shall extend, in case X is infinite - dimensional a rezult from [7] on a priori bounds of the solutions of problem (1.1) - (1.2). The growth condition with respect to u' that will be imposed to f is that from [18] .

In Section 4 we shall establish some results on the existence of solutions of the boundary value problem (1.1) - (1.2). Under several aspects, our results generalize those obtained in [7] and [18] .

In what is to follow, we let $\|u\| = \max(|u(t)| : t \in I)$ for $u \in C = C(I;X)$, $\|u\|_1 = \max(\|u\|, \|u'\|)$ for $u \in C^1 = C^1(I;X)$ and $\|u\|_2 = \max(\|u\|, \|u'\|, \|u''\|)$ for $u \in C^2 = C^2(I;X)$. Also , by J we denote the <u>duality mapping</u> of X.

2. LERAY-SCHAUDER'S ALTERNATIVE FOR CONDENSING MAPPINGS.

Let Y be a closed convex subset of the real Banach space X, Z an arbitrary subset of X and let $F : Z \to Y$ be a continuous mapping. Denote by α the Kuratowski's measure of noncompactness.

The mapping F is said to be <u>(α, ρ) - Lipschitz</u> (where $\rho > 0$) if for every bounded subset A of Z, $F(A)$ is bounded and

$$\alpha(F(A)) \leq \rho\alpha(A)$$

F is <u>α- Lipschitz</u> if there exists $\rho > 0$ such that F be (α, ρ) - Lipschitz.

F is said to be <u>condensing</u> if for every bounded subset A of Z , $F(A)$ is bounded and if $\alpha(A) > 0$, then

$$\alpha(F(A)) < \alpha(A) .$$

The fixed point theorem of D.N.Sadovskii [16] can be formulated as follows.

Theorem 2.1 (B.N.Sadovskii). If X is a real Banach space Y a closed convex subset of X and $F : Y \to Y$ a condensing mapping with $F(Y)$ bounded, then there exists $x \in Y$ such that $Fx = x$.

Proof. $\overline{CO}\, F(Y)$ is a closed bounded convex subset of Y and $F(\overline{CO}\, F(Y)) \subset \overline{CO}\, F(Y)$. Thus, we may apply to $F : \overline{CO}\, F(Y) \to \overline{CO}\, F(Y)$ the Sadovskii's fixed point theorem.

Let U be an open bounded subset of Y and let $\mathcal{A}(\bar{U}; Y)$ be the set of all condensing mappings $F : \bar{U} \to Y$ which are fixed point free on the boundary ∂U of U. A mapping F in $\mathcal{A}(\bar{U};Y)$ is called admissible.

Let $F \in \mathcal{A}(\bar{U};Y)$. F is inessential if there exists a fixed point free mapping $G \in \mathcal{A}(\bar{U};Y)$ such that the restrictions of F and G to ∂U coincide, i.e., $F|_{\partial U} = G|_{\partial U}$. If F is not inessential it is called essential.

It is clear that an admissible mapping F is essential if and only if each admissible mapping which coincides with F on ∂U, has at least one fixed point in U.

Lemma 2.1. Let $x_o \in U$. The mapping $F : \bar{U} \to Y$, $Fx = x_o$ for all $x \in \bar{U}$ is essential.

Proof. Since $\alpha(\{x_o\}) = 0$ it is clear that F is condensing. Let $G \in \mathcal{A}(\bar{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$. Define $H : Y \to Y$,

$$\begin{aligned} Hx &= x_o \ , \quad \text{if } x \in Y \setminus \bar{U} \\ &= Gx \ , \quad \text{if } x \in \bar{U} \ . \end{aligned}$$

Obviously, H is continuous. Also, if $A \subset Y$ is bounded, since $H(A) \subset G(A \cap \bar{U}) \cup \{x_o\}$, we see that $H(A)$ is bounded too. Now assume $\alpha(A) > 0$. In case $\alpha(A \cap \bar{U}) > 0$ we have

$$\begin{aligned} \alpha(H(A)) &\leq \alpha(G(A \cap \bar{U}) \cup \{x_o\}) = \alpha(G(A \cap \bar{U})) < \\ &< \alpha(A \cap \bar{U}) \leq \alpha(A) \ , \end{aligned}$$

whence $\alpha(H(A)) < \alpha(A)$. If $\alpha(A\cap\bar{U}) = 0$, then

$$\alpha(H(A)) \leq \alpha(G(A\cap\bar{U})) = 0 < \alpha(A) .$$

Thus, if $\alpha(A)>0$ then $\alpha(H(A)) < \alpha(A)$. Therefore, H is condensing. In addition, since $H(Y) = G(\bar{U}) \cup \{x_0\}$, $H(Y)$ is bounded. Hence we may apply Theorem 2.1. In consequence, there exists $x \in Y$ such that $Hx = x$. It is clear that $x \in U$ and so $Gx = x$. Therefore, F is essential.

Let $F,G \in \mathcal{A}(\bar{U};Y)$. F and G are called <u>homotopic</u> if there exists $H : I \times \bar{U} \to Y$ such that $H_\lambda = H(\lambda, \cdot) \in \mathcal{A}(\bar{U};Y)$ for all $\lambda \in I$, $H_0 = G$, $H_1 = F$ and $H(\cdot,x) : I \to Y$ is continuous uniformly with respect to $x \in \bar{U}$. If F and G are homotopic we write $F \overset{H}{\sim} G$.

<u>Lemma 2.2</u>. Let $F \in \mathcal{A}(\bar{U};Y)$. F is inessential if and only if it is homotopic to a fixed point free admissible mapping.

<u>Proof</u>. Suppose first that F is inessential. Then, there exists a fixed point free mapping $G \in \mathcal{A}(\bar{U};Y)$ such that $G|_{\partial U} = F|_{\partial U}$. It is no difficult to see (use properties of measures of noncompactness [1, pp.7]) that $H_\lambda = \lambda F + (1-\lambda) G \in \mathcal{A}(\bar{U};Y)$ and since

$$|H(\lambda_1,x) - H(\lambda_2,x)| \leq (|F(x)| + |G(x)|)\, |\lambda_1 - \lambda_2| \quad (x \in \bar{U}),$$

that $H(\cdot,x)$ is continuous uniformly with respect to $x \in \bar{U}$. Consequently, $F \overset{H}{\sim} G$.

Conversely, if $F \overset{H}{\sim} G$ where G is a fixed point free admissible mapping, we shall prove that H_λ is inessential for each $\lambda \in I$ whence, in particular, $F = H_1$ is inessential. For this, let $V = \{x \in \bar{U};$ there is $\lambda \in I$ with $H_\lambda x = x\}$. If $V = \emptyset$, then $H_1 = F$ has no fixed point in $\bar{U}$ and so F is inessential. Next, let us assume $V \neq \emptyset$. Taking into account that $H(\cdot,x)$ is continuous uniformly with respect to $x \in \bar{U}$ we easily see that V is closed. In additinn $V \cap \partial U = \emptyset$. By Urysohn's Theorem there is a continuous function $\theta : \bar{U} \to I$ such that $\theta(x) = 1$ for $x \in \partial U$ and $\theta(x) = 0$ for $x \in V$. Define $H^{*}_{\lambda} : \bar{U} \to Y$, $H^{*}_{\lambda} x = H_{\theta(x)\lambda}\, x$ for $x \in \bar{U}$ ($\lambda \in I$). Obviously H^{*}_{λ} is continuous. We now show that $H^{*}_{\lambda}(\bar{U})$ is bounded. Indeed, if it is no bounded, then there is a sequence $(x_n) \subset \bar{U}$ such that $|H^{*}_{\lambda} x_n| \to \infty$ as $n \to \infty$. Passing if necessary to a subsequence, we may assume that $\theta(x_n) \to \theta_0$ as $n \to \infty$, where $\theta_0 \in I$. Then, since $H(\cdot,x)$ is continuous uniformly with respect to $x \in \bar{U}$, we must have

$$|H(\theta(x_n)\lambda, x_n) - H(\theta_0\lambda, x_n)| \leq 1$$

for n sufficiently large. Next, from

$$|H^{*}_{\lambda} x_n| = |H(\theta(x_n)\lambda, x_n)| \leq |H(\theta(x_n)\lambda, x_n) - H(\theta_0\lambda, x_n)| + \\ + |H(\theta_0\lambda, x_n)|$$

we found that $|H(\theta_0\lambda, x_n)| \to \infty$ as $n \to \infty$.

Thus we arrive at a contradiction, because $H_{\theta_0\lambda}(\bar{U})$ is bounded. It follows that $H^{*}_{\lambda}(\bar{U})$ is bounded.

Let $A \subset \bar{U}$ with $\alpha(A) > 0$. We will show that $\alpha(H^{*}_{\lambda}(A)) < \alpha(A)$. Let $\varepsilon(\lambda) > 0$ such that $\alpha(H_{\lambda}(A)) < \alpha(A) - 3\,\varepsilon(\lambda)$ and let $V(\lambda)$ be a neighbourhood of λ such that $|H_{\mu} x - H_{\lambda} x| \leq \varepsilon(\lambda)$ for all $\mu \in V(\lambda)$ and $x \in A$. Thus, the set $H(V(\lambda), A)$ can be covered by a finite number of sets of diameter at most equal with $\alpha(A) - 3\cdot\varepsilon(\lambda) + 2\cdot\varepsilon(\lambda) = \alpha(A) - \varepsilon(\lambda)$. Hence $\alpha(H(V(\lambda),A)) < \alpha(A)$. Consider $\{V(\lambda_i);\ i = 1,\dots,n\} \subset \{V(\lambda);\ \lambda \in I\}$ a finite covering of the compact interval I. Then $\alpha(H(I,A)) \leq \max\,(\alpha(H(V(\lambda_i),A))\ ;\ i = 1,\dots,n) < \alpha(A)$, whence $\alpha(H^{*}_{\lambda}(A)) < \alpha(A)$. Thus we have shown that $H^{*}_{\lambda} \in \mathcal{A}(\bar{U};Y)$. Further, it is easy to see that H^{*}_{λ} is a fixed point free mapping and $H^*_\lambda|_{\partial U} = H_\lambda|_{\partial U}$. It follows that H_λ is inessential.

Now we can prove the Leray - Schauder's alternative for condensing mappings.

Theorem 2.2. Let $F, G \in \mathcal{A}(\bar{U};Y)$ be two homotopic mappings. Then F is essential if and only if G is essential.

Proof. Apply Lemma 2.2 and use the transitivity of the homotopy relation.

Remark 2.1. The classical Leray - Schauder's alternative can be deduced from Theorem 2.2 if one considers: $Y = X$, U the unit ball of X, $H : I \times \bar{U} \to X$ a completely continuous mapping satisfying $H(\lambda, x) \neq x$ whenever $\lambda \in I$ and $x \in \partial U$, $H(0,x) = 0$ for all $x \in \bar{U}$ and $H(1,\cdot) = F$.

Remark 2.2. Theorem 2.2 remains true if the condensing mappings are defined with respect to certain other measures of noncompactness.

3. A PRIORI BOUNDS ON SOLUTIONS.

Lemma 3.1. Let $f : I \times X \times X \times C(I;X) \to X$ be a continuous mapping such that for each $x \in X$ satisfying $\|x\| > M \geq 0$ there is $x^* \in Jx$ for which

$$(3.1) \qquad (x^*, f(t,x,y;z)) > 0$$

whenever $t \in I$, $y \in X$ with $(x^*, y) = 0$ and $z \in C^2(I;X)$ with $\|z\| = \|x\|$.

If $u \in C^2(I;X)$ is a solution to (1.1) for which there exists $t_o \in [0,1]$ such that $\|u\| = \|u(t_o)\|$ and $(x^*_o, u'(t_o)) = 0$ for every $x^*_o \in Ju(t_o)$, then

$$(3.2) \qquad \|u\| \leq M .$$

Proof. If we suppose that $\|u(t_o)\| > M$ then, by (3.1) there is $x^*_o \in Ju(t_o)$ such that

$$(x_0^*, f(t_0, u(t_0), u'(t_0); u)) > 0 .$$

Since f is continuous, it follows that there is $\delta > 0$ such that

$$(x_0^*, f(t_0+h, u(t_0+h), u'(t_0+h); u)) > 0$$

whenever $|h| < \delta$ and $t_0 + h \in I$. Hence

$$(x_0^*, u''(t_0+h)) > 0 \quad (|h| < \delta, t_0 + h \in I) ,$$

whence, using Taylor's formula

$$u(t_0+h) - u(t_0) = h u'(t_0) + (h^2/2)u''(t_0 + sh) ,$$

where $s = s(h) \in I$, we deduce that

$$(x_0^*, u(t_0 + h) - u(t_0)) > 0 .$$

On the other hand, since $x_0^* \in Ju(t_0)$ we must have

$$(x_0^*, u(t_0+h) - u(t_0)) \leqslant \frac{1}{2}|u(t_0+h)|^2 - \frac{1}{2}|u(t_0)|^2 \leqslant 0 ,$$

which is a contradiction. Therefore, (3.2) must hold.

We shall obtain bounds on the derivatives of solutions to (1.1) by using the following result.

Lemma 3.2 (K.Schmitt, R.Thompson [18]). Suppose that

(i) There is $M \geqslant 0$ such that $\|u\| \leqslant M$ for every solution u to (1.1) ;

(ii) There is a nondecreasing function $\Psi : [0, +\infty[\to]0, +\infty[$ such that

$$\liminf_{t \to +\infty} t^2/\Psi(t) > 4M \tag{3.4}$$

and

$$|f(t,x,y;z)| \leqslant \Psi(|y|) \tag{3.5}$$

for all $t \in I$, $x,y,z \in X$ with $|x| \leqslant |z| \leqslant M$.

Then there is a constant M_1 such that

$$\|u'\| \leqslant M_1 \tag{3.6}$$

for each solution u to (1.1) .

4. EXISTENCE THEOREMS.

Consider the boundary value problem

(4.1) $$Lu = g(t,u,u';u) \qquad (t \in I),$$

(4.2) $$W_i u = V_i u \qquad (i = 1,2),$$

where $Lu = u'' + b(t)u' + c(t)u$, $b,c \in C(I;X)$.

Let $C_0^2 = \{u \in C^2(I;X) \ ; \ W_i u = 0, \ i = 1,2\}$. With the norm $\|\cdot\|_2$, C_0^2 is a Banach space.

If the mapping $L : C_0^2 \to C$ has an inverse, then $L_1 : C^2 \to C \times X^2$, $L_1 u = (Lu, W_1 u, W_2 u)$ has also an inverse and L_1^{-1} is a bounded linear mapping.

In what is to follow, we assume that $g : I \times X \times X \times C(I;X) \to X$ and $\varphi, \psi : X^4 \to X$ are continuous and we denote

$$G : C^2 \to C \times X^2 \ , \ Gu = (g(\cdot,u,u';u), \ V_1 u, \ V_2 u).$$

Obviously G is continuous.

<u>Theorem 4.1.</u> If $L : C_0^2 \to C$ has an inverse, $L_1^{-1}G : C^2 \to C^2$ is condensing and there is $M > 0$ such that $\|u\|_2 < M$ for each solution u to the boundary value problem

(4.3) $$Lu = \lambda g(t,u,u';u) \qquad (t \in I),$$

(4.4) $$W_i u = \lambda V_i u \qquad (i = 1,2),$$

for each $\lambda \in I$, then the problem (1.1) - (1.2) has at least one solution.

<u>Proof.</u> Apply Theorem 2.2 to $L_1^{-1}G$ and to the zero mapping, where $U = \{u \in C^2 \ ; \ \|u\|_2 < M\}$. By Lemma 2.1 the zero mapping is essential. Hence $L_1^{-1}G$ is essential too. Thus, there is $u \in U$ such that $L_1^{-1}G u = u$, that is u is a solution to (1.1) - (1.2).

<u>Remark 4.1.</u> If $V_i \equiv 0$, $i = 1,2$, then $G : C^2 \to C \times \{0\} \times \{0\} \equiv C$ and in Theorem 4.1 we may take instead of $L_1^{-1}G$

the mapping $L^{-1}G : C^2_\sigma \to C^2_\sigma$.

Remark 4.2. If $V_1 \equiv r$ and $V_2 \equiv s$, where r and s are fixed elements in X, then in Theorem 4.1 we may take instead of $L_1^{-1}G$ the mapping $N^{-1}G : C^2_b \to C^2_b$, where $C^2_b = \{u \in C^2 ; W_1 u = r, W_2 u = s\}$, $N : C^2_b \to C$, $Nu = L(u-u_b)$ and u_b is the unique solution to the boundary value problem $Lu = 0$, $W_1 u = r$, $W_2 u = s$. Then $N^{-1}u = L^{-1}u + u_b$ and Theorem 4.1 can be formulated as follows : if $L^{-1}G$ is condensing and $\|u\|_2 < M + \|u_b\|_2$ for each solution u to (4.3) - (4.4) ($\lambda \in I$), then there is at least one solution u to (1.1) satisfying $W_1 u = r$ and $W_2 u = s$.

We now consider the boundary conditions (1.2) in the following cases :

(a) $a > 0$, $c > 0$, φ and ψ bounded ;

(b) $a = 0$, $c > 0$, $\varphi \equiv 0$ and ψ bounded
($a > 0$, $c = 0$, φ bounded and $\psi \equiv 0$) ;

(c) $a > 0$, $d > 0$, $\varphi(x_1,x_2,x_3,x_4) = a\,x_3 - b\,x_2$
and $\psi(x_1,x_2,x_3,x_4) = c\,x_3 + d\,x_2$
(periodic boundary conditions) ;

(d) $a > 0$, $c > 0$, $(x_1^*, \varphi(x_1,x_2,x_3,x_4)) \leqslant 0$ and $(x_3^*, \psi(x_1,x_2,x_3,x_4)) \leqslant 0$
for all $x_1,x_2,x_3,x_4 \in X$, $x_1^* \in Jx_1$ and $x_3^* \in Jx_3$;

(e) $a > 0$, $c = 0$, $(x_1^*, \varphi(x_1,x_2,x_3,x_4)) \leqslant 0$, $\psi \equiv 0$
($a = 0$, $c > 0$, $\varphi \equiv 0$, $(x_3^*, \psi(x_1,x_2,x_3,x_4)) \leqslant 0$)
for all $x_1,x_2,x_3,x_4 \in X$, $x_1^* \in Jx_1$ ($x_3^* \in Jx_3$).

In all these cases the mapping $L : C^2_o \to C$, $Lu = u''$ has an inverse.

Theorem 4.2. Let f, φ and ψ be (α, ρ) - Lipschitz. Suppose that

(i) For each $x \in X$ satisfying $|x| > M \geqslant 0$ there is $x^* \in Jx$ for which (3.1) holds ;

(ii) There exists a nondecreasing function $\Psi : [0,+\infty[\longrightarrow]0,+\infty[$ such that

(4.5) $$\lim_{t \to +\infty} t^2/\Psi(t) = +\infty,$$

and

(4.6) $$|f(t,x,y;z)| \leqslant \Psi(|y|)$$

for all $t \in I$, $x,y,z \in X$ with $|x| \leqslant |z|$;

(iii) $\|L_1^{-1}\| \rho < 1$.

Then the boundary value problem (1.1) - (1.2) has at least one solution in each of cases (a) - (e) .

Proof. By (iii) the mapping $L_1^{-1}G$ is condensing. Thus, with a view to apply Theorem 4.1 we have only to prove the boundedness with respect to $\|\cdot\|_2$ of the set of solutions to (4.3) - (4.4). We shall use Lemma 3.1 and Lemma 3.2. Let u be a nonzero solution to equation $u'' = \lambda f(t,u,u';u)$ $(\lambda \in I)$ satisfying (4.4). Let t_0 be such that $\|u\| = |u(t_0)|$. Suppose first that $t_0 \in]0,1[$. Then for every $x_0^* \in Ju(t_0)$ we have

$$(x_0^*, u(t) - u(t_0)) \leqslant \tfrac{1}{2}|u(t)|^2 - \tfrac{1}{2}|u(t_0)|^2 \leqslant 0 ,$$

whence for $t \uparrow t_0$ we get $(x_0^*, u'(t_0)) \geqslant 0$ and for $t \downarrow t_0$, $(x_0^*, u'(t_0)) \leqslant 0$. Therefore $(x_0^*, u'(t_0)) = 0$. Thus, we may apply Lemma 3.1 to conclude that $\|u\| \leqslant M$ for each solution u to (4.3) - (4.4). Further, by Lemma 3.2 we have $\|u'\| \leqslant M_1$. Consequently, since f carries bounded sets into bounded sets, we also obtain $\|u''\| \leqslant M_2$. So it remains only to investigate the case $t_0 \in \{0,1\}$. In case :

(a) If $t_o = 0$, then by $(x_o^*, u(t) - u(0)) \leq \frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(0)|^2 \leq 0$, for $x_o^* \in Ju(0)$, we deduce $(x_o^*, u'(0)) \leq 0$. Hence

$$0 \geq b(x_o^*, u'(0)) = a\,(x_o^*, u(0)) - \lambda\,(x_o^*, V_1 u) \geq$$
$$\geq a|u(0)|^2 - r|u(0)|, \quad \text{where } |\varphi| \leq r.$$

It follows that $|u(0)| \leq r/a$. Similarly, if $t_o = 1$, we get $|u(1)| \leq s/c$, where $|\psi| \leq s$. Therefore,

$$\|u\| \leq \max\,(M, r/a, s/c).$$

(b) If $t_o = 0$ then since $u'(0) = 0$ we may apply Lemma 3.1 and we get $\|u\| \leq M$. If $t_o = 1$ then, as in case (a) we obtain $\|u\| \leq s/c$. Therefore, $\|u\| \leq \max\,(M, s/c)$. (For $a > 0$, $c = 0$, φ bounded and $\psi \equiv 0$ we have $\|u\| \leq \max\,(M, r/a)$).

(c) Since $u(0) = u(1)$, $|u(0)| = \|u\|$, we have

$$(x_o^*, u(t) - u(0)) = (x_o^*, u(t) - u(1)) \leq 0$$

for each $x_o^* \in Ju(0)$. Whence

$$(x_o^*, u'(0)) \leq 0 \text{ and } (x_o^*, u'(1)) \geq 0.$$

But since $u'(0) = u'(1)$ it follows $(x_o^*, u'(0)) = 0$, which permits us to apply Lemma 3.1.

(d) We will show that in this case $t_o \in\,]0,1[$. To this end let us first assume $b = 0$. Then, since $a\,u(0) = \lambda\,\varphi(u(0), u'(0), u(1), u'(1))$ it follows $(x_o^*, u(0)) \leq 0$ for all $x_o^* \in Ju(0)$, whence $|u(0)| = 0$. Thus $t_o \neq 0$. Next let $b > 0$ and suppose that $|u(o)| = \|u\|$. Then $(x_o^*, u'(0)) \leq 0$ for all $x_o^* \in Ju(0)$. On the other hand, since $(x_o^*, V_1 u) \leq 0$ we have

$$(4.7) \qquad (x_o^*, u'(0)) = \frac{a}{b}|u(0)|^2 - \frac{\lambda}{b}\,(x_o^*, V_1 u) \geq 0.$$

In consequence, $(x_o^*, u'(0)) = 0$ and since $a > 0$, by (4.7) we also deduce $|u(0)| = 0$, a contradiction. So $t_o \neq 0$. Similarly, using the second boundary condition in (4.4) one shows that $t_o \neq 1$.

(e) As in case (d) it is shown that $t_o \neq 0$. If $t_o = 1$, then $u'(1) = 0$ and we may apply Lemma 3.1.

Remark 4.3. If $V_1 \equiv r$ and $V_2 \equiv s$, then in Theorem 4.2 instead of (iii) we may require that

$$\| L^{-1} \| \rho < 1 \ ,$$

where L^{-1} is the integral operator having as kernel the Green's function associated to $Lu = u''$ and $W_i u = 0$ $(i = 1,2)$ (see Remarks 4.1 and 4.2).

In particular, condition (iii) in Theorem 4.2 is satisfied if f, φ and ψ are completely continuous ($\rho = 0$).

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