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ON THE QUASICONVEX FUNCTIONS OF HIGHER ORDER

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Let n be a nonnegative integer. Consider n+3 real numbers

(1) $x_1 < x_2 < \dots < x_{n+3}$

and a real function f defined on the points (1). Denote by $D_i(f)$ the divided difference

$$[x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+3}; f],$$

where $1 \le i \le n+3$. By the mean value theorem of divided differences, we have

(2) $(x_{n+3}-x_1)D_i(f) = (x_i-x_1)D_{n+3}(f) + (x_{n+3}-x_i)D_1(f),$

for each i, $1 \le i \le n+3$ (see [2], p.163). Whence, we obtain

(3) $$D_k(f)-D_i(f) = \frac{x_k-x_i}{x_{n+3}-x_1}(D_{n+3}(f)-D_1(f)),$$

for all i, k such that $1 \le i, k \le n+3$.

PROPOSITION 1. <u>The following statements are equivalent:</u>

1^o. <u>We have</u>

(4) $[x_2, x_3, \dots, x_{n+2}; f] \le \max([x_1, x_2, \dots, x_{n+1}; f],$
$[x_3, x_4, \dots, x_{n+3}; f]).$

2°. <u>We have</u>

$$(5) \qquad 0 \leqslant \max\left(-[x_1,x_2,\ldots,x_{n+2};f],[x_2,x_3,\ldots,x_{n+3};f]\right).$$

3°. <u>For every i and k with $1 \leqslant i < k \leqslant n+3$, $k-i \geqslant 2$, we have</u>

$$(6) \qquad 0 \leqslant \max\left(-[x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+3};\ f],\ [x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3};\ f]\right).$$

<u>Proof</u>. The equivalence $1^{\circ} \Longleftrightarrow 2^{\circ}$ follows imediately by the recurrence formula of divided differences (see [4], p.8).

The implication $3^{\circ} \Longrightarrow 2^{\circ}$ is obvious, if we take in (6) $i = 1$ and $k = n+3$.

Assume now that 3° is not satisfied, i.e., there exists i and k, $1 \leqslant i < k \leqslant n+3$, $k-i \geqslant 2$, such that $D_i(f) < 0$ and $D_k(f) > 0$. Then, by (3), we have $D_{n+3}(f) - D_1(f) > 0$. Consequently, again by (3), we obtain

$$D_i(f) - D_1(f) > 0 \quad \text{and} \quad D_{n+3}(f) - D_k(f) > 0 ,$$

whence $D_1(f) < 0$ and $D_{n+3}(f) > 0$, which shows that 2° is not satisfied too. Thus, $2^{\circ} \Longrightarrow 3^{\circ}$, which completes the proof.

The following proposition can be proved similarly.

PROPOSITION 2. <u>The following statements are equivalent</u>:

1°. <u>We have</u>

$$(4') \qquad [x_2,x_3,\ldots,x_{n+2};f] < \max\left([x_1,x_2,\ldots,x_{n+1};f],\ [x_3,x_4,\ldots,x_{n+3};f]\right).$$

2°. <u>We have</u>

$$(5') \qquad 0 < \max\left(-[x_1,x_2,\ldots,x_{n+2};f],[x_2,x_3,\ldots,x_{n+3};f]\right).$$

3°. <u>For every i and k with $1 \leqslant i < k \leqslant n+3$, $k-i \geqslant 2$, we have</u>

$$(6') \qquad 0 < \max\left(-[x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+3};f],\ [x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3};f]\right).$$

Let X be a set of real numbers containing at least $n+3$ elements and let f be a real function defined on X.

The functions f satisfying condition (4) ((4')) , for every system of points (1) in X, have been first considered by Elena Popoviciu in [1]. They have been called quasiconvex functions of order n (strictly quasiconvex functions of order n). These functions represent a natural generalization of nonconcave (convex) functions of order n.

For $n = 0$, the functions f satisfying condition (4),i.e.,

$$f(x_2) \leq \max (f(x_1),f(x_3)) \quad ,$$

for every system of points $x_1 < x_2 < x_3$ in X, had been first studied by T. Popoviciu [3] , but not under the name of quasiconvex functions (see also [4] , p.22) .

According to Proposition 1 (Proposition 2), each of inequalities (4),(5),(6),((4'),(5'),(6')) may be used in the definition of quasiconvex (strictly quasiconvex) functions of order n.

Let us remark that we can speak about quasiconvex (strictly quasiconvex) functions of order n, even for $n=-1$, if we use in definition inequality (5)((5')) instead of (4)((4')).

PROPOSITION 3. *Let $f : X \to \mathbb{R}$ be a quasiconvex function of order $n, n \geq 0$ and let the points (1) in X satisfy*

(7) $$[x_1,x_2,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+3}; f] = 0 \quad ,$$

for some j , $2 \leq j \leq n+2$. Then

a) *We have*

(8) $$0 \leq [x_1,x_2,\ldots,x_{n+3} ; f] \; .$$

b) *We have*

(9) $$[x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+3};f] \leq 0 \leq [x_1,\ldots,x_{i-1}, x_{i+1},\ldots,x_{n+3} ; f],$$

whenever $1 \leq i < j < k \leq n+3$.

Proof. The statements a) and b) being equivalent, as follows immediately by (3) and (7), we have only to prove that b) is true. To this end, assume that $D_j(f) = 0$ for a certain j, $2 \leq j \leq n+2$ and take arbitrary i and k with $1 \leq i < j < k \leq n+3$. Applying (2) to j instead of i, we see that $D_1(f)D_{n+3}(f) \leq 0$. On the other hand, since f is quasiconvex of order n, we have (5), that is $0 \leq \max(-D_{n+3}(f), D_1(f))$. Consequently, $D_1(f) \geq 0$ and $D_{n+3}(f) \leq 0$, whence, using (3), we conclude that $D_k(f) - D_j(f) \leq 0$ and $D_j(f) - D_i(f) \leq 0$. Thus, we have (9), which completes the proof.

Similarly we can prove the following proposition.

PROPOSITION 4. Let $f : X \to \mathbb{R}$ be a strictly quasiconvex function of order n, $n \geq 0$ and let the points (1) in X satisfy condition (7) for some j, $2 \leq j \leq n+2$. Then

a) We have

(8') $$0 < [x_1, x_2, \ldots, x_{n+3} ; f] .$$

b) We have

(9') $$[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+3}; f] < 0 < [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+3} ; f] ,$$

whenever $1 \leq i < j < k \leq n+3$.

In the paper [5] (see also [6]) the functions f satisfying (8') for every system (1) of points satisfying (7), have been called $(\mathcal{P}_0, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$ - quasiconvex and the functions f for which assertion 3^0 in Proposition 2 is true, have been called strongly $(\mathcal{P}_0, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$ - quasiconvex functions. These functions have been defined in connection with the notion of decomposition of an interpolation operator (see [6]).

We state now the main results.

THEOREM 1. Let $f : I \to R$ be a continuous function, where I is an interval of real numbers. The following statements are equivalent :

1°. The function f is quasiconvex of order n.

2°. We have

(10) $$0 \leq [x_1, x_2, \ldots, x_{n+3} ; f]$$

for every system (1) of points in $\bar{I}$ satisfying

$$[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+3}; f] = 0 ,$$

for some j, $2 \leq j \leq n+2$.

3°. We have

(11) $$[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+3}; f] \leq 0 \leq [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+3} ; f]$$

for every system (1) of points in I, provided that $1 \leq i < j < k \leq \leq n+3$ and

$$[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+3} ; f] = 0 .$$

Proof. According to Proposition 3, we have only to prove that 3° $\Rightarrow$ 1°. Assume that 1° is not true. Then, there exists a system (1) of points in I such that (5) is not satisfied. Hence $D_1(f) < 0$ and $D_{n+3}(f) > 0$.

Case 1 : There exists j, $2 \leq j \leq n+2$ such that $D_j(f) = 0$. Then, by (11), we have $D_{n+3}(f) \leq 0 \leq D_1(f)$, a contradiction . In this case the proof is finished.

Case 2 : There is i, $1 \leq i \leq n+2$, such that $D_i(f) < 0 < D_{i+1}(f)$. Then, by the continuity of f, there exists c, $x_i < c < x_{i+1}$, such that

(12) $$[x_1, \ldots, x_{i-1}, c, x_{i+2}, \ldots, x_{n+3}; f] = 0 .$$

We consider the following subcases :

a) $i = 1$. Replacing in (1) x_1 by c we shall obtain for the new system of points : $D_1(f) < 0$ and, by (12), $D_2(f)=0$, which contradicts (11).

b) $i = n+2$. Replacing in (1) x_{n+3} by c we shall obtain for the new system of points : $D_{n+3}(f) > 0$ and, by (12), $D_{n+2}(f) = 0$, which also contradicts (11).

c) $2 \leq i \leq n+1$. Denote by $D^c_{i,k}(f)$ the divided diffe - rence of f on the points : $c, x_1, x_2, \ldots, x_{n+3}$, except x_i and x_k. We will show that

(13) $$D^c_{1,i}(f) < 0 \quad \text{or} \quad D^c_{i+1,n+3}(f) > 0 .$$

Suppose, a contrario, that $D^c_{1,i}(f) \geq 0$ and $D^c_{i+1,n+3}(f) \leq 0$. We will derive a contradiction. Indeed, by $D^c_{1,i}(f) \geq 0$ and $D_1(f) < 0$, we must have

$$[x_2 \ldots, x_i, c, x_{i+1}, \ldots, x_{n+2}; f] < 0 ,$$

as follows if we apply (3) to the points : $x_2, \ldots, x_i, c, x_{i+1}, \ldots, x_{n+3}$.

Similarly, by $D^c_{i+1,n+3}(f) \leq 0$ and $D_{n+3}(f) > 0$, we must have

$$[x_2 \ldots, x_i, c, x_{i+1}, \ldots, x_{n+2}; f] > 0 ,$$

a contradiction. Thus, (13) holds.

Now, if $D^c_{1,i}(f) < 0$, then, replacing in (1) x_i by c, we shall obtain for the new system of $n+3$ points : $D_1(f) < 0$ and, by (12), $D_{i+1}(f) = 0$, which contradicts (11). Also, if $D^c_{i+1,n+3}(f) > 0$, then, replacing in (1) x_{i+1} by c, we shall ob - tain for this new system of $n+3$ points : $D_{n+3}(f) > 0$ and , by (12), $D_i(f) = 0$, a contradiction to (11).

Therefore, $3^o \Longrightarrow 1^o$ as desired.

THEOREM 2. Let $f : I \longrightarrow \mathbb{R}$ be a continuous function, where I is an interval of real numbers. The following statements are equivalent:

1^o. The function f is strictly quasiconvex of order n.

2^o. We have

(10') $$0<[x_1,x_2,\dots,x_{n+3};f]$$

for every system (1) of points in I satisfying

$$[x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+3};f]=0,$$

for some j, $2\le j\le n+2$.

3^o. We have

(11') $$[x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+3};f]<0<[x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+3};f]$$

for every system (1) of points in I, provided that $1\le i<j<k\le n+3$ and

$$[x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+3};f]=0.$$

The proof of Theorem 2 is similar to that of Theorem 1.

In the particular case, n=0, Theorem 1 and Theorem 2 have been given by Elena Popoviciu [1].

R E F E R E N C E S

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This paper is in final form and no version of it will be submitted for publication elsewhere.