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TOPOLOGICAL TRANSVERSALITY, PERTURBATION THEOREMS AND SECOND ORDER DIFFERENTIAL EQUATIONS

by

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1. Abstract. The topological transversality theorem for condensing mappings stated in [7] is used to prove some perturbation theorems: a theorem on γ-condensing perturbations of hyperaccretive mappings and a Browder type result on the perturbation of some bijective mappings by γ-Lipschitz mappings. An application concerning the existence and the uniqueness of solution to a boundary value problem for nonlinear second order diffential equations in Banach spaces is finally given.

2. Preliminaries. Let X be a real Banach space, X^* its dual. Denote both the norm in X and its dual norm in X^* by $|\cdot|$. The value of $x^*\in X^*$ at $x\in X$ is denoted by (x^*,x). In case $X=\mathbb{R}^n$ the bilinear functional $(.,.)$ stands for the scalar product.

Let $\mathcal{F}$ be the duality mapping of X, i.e. $\mathcal{F}:X\to 2^{X^*}$,
$\mathcal{F}x=\left\{x^*\in X^* : (x^*,x)=|x|^2=|x^*|^2\right\}$ and let $(.,.)_+$ be the semi-inner product on X defined by

$$(x,y)_+=|y|\lim_{t\downarrow 0} t^{-1}(|y+tx|-|y|).$$

or equivalently $(x,y)_+ = \sup\{(y^*,x) : y^* \in \mathcal{F}y\}$.

A mapping $F : D \to 2^X \setminus \{\emptyset\}$, $D \subset X$, is said to be accretive if $(y_1-y_2, x_1-x_2)_+ \geq 0$ for all $x_1, x_2 \in D$, $y_1 \in Fx_1$ and $y_2 \in Fx_2$. An accretive mapping F is said to be hyperaccretive if $I+sF$ is onto X for some (equivalently for all) $s>0$. Recall that if F is hyperaccretive then the mapping $R_s=(I+sF)^{-1}:X \to D$ is nonexpansive for each $s>0$ and $R_s x \to x$ as $s \downarrow 0$ for all $x \in \bar{D}$ (see [3], pp.126).

Let γ denote α or β, Kuratowski's or the ball measure of noncompactness; for each bounded subset B of a metric space one has

$\alpha(B)=\inf\{d>0 : B$ admits a finite cover by sets of diameter $\leq d\}$

and

$\beta(B)=\inf\{r>0 : B$ can be covered by finitely many balls of radius $r\}$

Clearly, $\beta(B) \leq \alpha(B) \leq 2\beta(B)$.

A continuous mapping $F : Y \to X$ (Y being a metric space) is called γ-Lipschitz if $\gamma(F(B)) \leq k\,\gamma(B)$ for some $k \geq 0$ and all bounded $B \subset Y$. We write k-γ-Lipschitz if k is important. F is said to be γ-condensing if $\gamma(F(B)) < \gamma(B)$ whenever $B \subset Y$ is bounded and $\gamma(B)>0$.

Let $J=[0,1]$. Denote by C the real Banach space $C(J;X)$ with the norm $\|u\| = \max\{|u(t)| : t \in J\}$ and by C^n $(n \geq 1)$ the space $C^n(J;X)$ endowed with the norm $\|u\|_n = \max\{\|u^{(i)}\| : i=0,\dots,n\}$

We shall denote by γ_n and simply by γ, the corresponding measure of noncompactness on the space C^n, respectively on C. If $B \subset C$ is bounded and equicontinuous, then

$$\alpha(B) = \sup\{\alpha(B(t)) : t \in J\},$$

where $B(t) = \{u(t) : u \in B\}$ (see [3], Proposition 7.3 (a)).

3. The topological transversality theorem for γ-condensing mappings

Let X be a real Banach space, K a closed convex subset of X and let $U \subset K$ be bounded and open in K. Denote by $\bar{U}$ and ∂U the closure and the boundary of U in K. Let $\mathcal{A}_{\partial U}(\bar{U};K)$ be the set of all γ-condensing mappings $F : \bar{U} \to K$ with $x \neq Fx$ for any $x \in \partial U$. The elements of $\mathcal{A}_{\partial U}(\bar{U};K)$ are called admissible. A mapping $F \in \mathcal{A}_{\partial U}(\bar{U};K)$ is said to be essential if any admissible mapping G which coincides with F on ∂U, has a fixed point. An admissible mapping which is not essential is called inessential.

Proposition 3.1. The constant mapping $F : \bar{U} \to K$, $Fx = x_0$ for all $x \in \bar{U}$, where $x_0 \in U$, is essential.

Two admissible mappings F_0 and F_1 are said to be homotopic if there exists $H : J \times \bar{U} \to K$ such that $H(s,.) \in \mathcal{A}_{\partial U}(\bar{U};K)$ for all $s \in J$, $H(0,.) = F_0$, $H(1,.) = F_1$ and $\{H(.,x) : x \in \bar{U}\}$ is equicontinuous.

Theorem 3.2. Two homotopic admissible mappings are both essential or both inessential.

The proofs of Proposition 3.1 and Theorem 3.2 can be found in [7]. They reproduce with some specifical changes those of the similar results on completely continuous mappings (see [4]).

4. Perturbation theorems.

We will present two applications of Theorem 3.2.

Theorem 4.1. Let X be a real Banach space, $T : D \to 2^X \setminus \{\emptyset\}$, $D \subset X$, a hyperaccretive mapping, $U \subset X$ open bounded with $\bar{U} \subset D$ and $S : \bar{U} \to X$ γ-condensing. If $x - x_0 \notin s(Sx - Tx - x_0)$ for some $x_0 \in U$ and all $s \in]0,1]$, $x \in \partial U$, then there exists $x \in U$ such that $x \in (S-T)x$.

Proof. We shall apply Theorem 3.2. to $F_0, F_1 : \bar{U} \to X$, $F_0 = x_0$,

$F_1 = (I+T)^{-1}S$. To this end, define $H : J \times \overline{U} \longrightarrow X$, $H(s,x) = R_s((1-s)x_0+sSx)$. Since S is γ-condensing and R_s is nonexpansive, $H(s,.)$ is γ-condensing too. Also, the hypothesis on ∂U guaranties that $x \neq H(s,x)$ for all $s \in J$ and $x \in \partial U$. Hence, $H(s,.) \in \mathcal{A}_{\partial U}(\overline{U} ; X)$ for all $s \in J$. It remains only to show that $\{H(.,x) : x \in \overline{U}\}$ is equicontinuous at each $s_0 \in J$. For $s_0 = 0$ this follows from

$$|H(s,x) - x_0| \leq |R_s((1-s)x_0+sSx)-R_sx_0|+$$

$$+|R_sx_0-x_0| \leq s|Sx-x_0|+|R_sx_0-x_0| ,$$

because $S(\overline{U})$ is bounded and $R_sx_0 \longrightarrow x_0$ as $s \downarrow 0$. Now let $s_0 \in]0,1]$. At the beginning we show that $\{(I+(.)T)^{-1}x : x \in B\}$ is equicontinuous at s_0 whenever $B \subset X$ is bounded. Indeed, since $\frac{1}{s}(x-R_sx) \in TR_sx$ for any $s \in]0,1]$, we have

$$0 \leq (\frac{1}{s}(x-R_sx) - \frac{1}{s_0}(x-R_{s_0}x), R_sx-R_{s_0}x)_+ =$$

$$= ((\frac{1}{s} - \frac{1}{s_0})(x-R_{s_0}x) - \frac{1}{s}(R_sx-R_{s_0}x), R_sx-R_{s_0}x)_+ =$$

$$= ((\frac{1}{s} - \frac{1}{s_0})(x-R_{s_0}x), R_sx-R_{s_0}x)_+ - \frac{1}{s}\|R_sx-R_{s_0}x\|^2 .$$

It follows that $|R_sx-R_{s_0}x| \leq \frac{1}{s_0}|s-s_0||x-R_{s_0}x|$, whence the equicontinuity of $\{(I+(.)T)^{-1}x : x \in B\}$ at s_0 is immediate because B and $R_{s_0}(B)$ are bounded. Now, from

$$|H(s,x)-H(s_0,x)| \leq |R_s((1-s)x_0+sSx)-R_{s_0}((1-s)x_0+sSx)|+$$

$$+|R_{s_0}((1-s)x_0+sSx)-R_{s_0}((1-s_0)x_0+s_0Sx)| \leq$$

$$\leq |R_s((1-s)x_0+sSx)-R_{s_0}((1-s)x_0+sSx)|+|s-s_0||x_0-Sx|,$$

using the boundedness of $S(\bar{U})$ and the equicontinuity just proved for $B = \mathrm{conv}(\{x_0\} \cup S(\bar{U}))$, we may infer that $\{H(.,x): x \in \bar{U}\}$ is equicontinuous at s_0, as desired. Thus, F_0 and F_1 are homotopic and since by Proposition 3.1 F_0 is essential, it follows by Theorem 3.2 that F_1 is also essential. Consequently, F_1 has a fixed point $x \in U$, that is $x \in (S-T)x$.

<u>Theorem 4.2</u>. Let X be a real Banach space, Y a metric space, $T : Y \to X$ and $S : Y \to X$ two mappings such that the following conditions are satisfied:

(i) S is γ-Lipschitz;

(ii) T maps bounded sets into bounded sets;

(iii) For each $s \in J$, $T_s = T-sS$ is injective, T_s^{-1} is continuous on its domain and there exists $c_s > 0$ such that

$$(1) \qquad c_s\,\gamma(B) \leq \gamma(T_s(B))$$

whenever $B \subset Y$ is bounded;

(iv) $\{T_s^{-1}x : s \in J\}$ is bounded for each $x \in X$.

Then, if T is surjective, $T-S$ is surjective too.

<u>Proof</u>. At the beginning we will show that the constant c_s in (1) may be considered independent on s. To this end, let $s_0 \in J$ be arbitrar, $s \in J$ and c_s the greatest constant for which (1) holds. Let $B \subset Y$ be bounded. If $\varepsilon > \alpha(T_s(B))$, then B can be covered by finitely many subsets B_i^1 such that diam $T_s(B_i^1) \leq \varepsilon$ and, on the other hand, by (1), $\alpha(B) < \varepsilon/(2c_s)$. Then, S being γ-Lipschitz, say k-γ-Lipschitz, one has $\alpha(S(B)) \leq 2k\alpha(B) < k\varepsilon/c_s$. It follows that B also admits a finite cover by subsets B_i^2 with

diam $S(B_j^2) \leq k\varepsilon/c_s$. Now, for $y_1, y_2 \in B_i^1 \cap B_j^2$ one has

$$|T_{s_0}y_1 - T_{s_0}y_2| \leq |T_s y_1 - T_s y_2| + |s-s_0||Sy_1 - Sy_2| \leq$$

$$\leq \varepsilon + |s-s_0|k\varepsilon/c_s .$$

Thus, the sets $T_{s_0}(B_i^1 \cap B_j^2)$ represent a finite cover of $T_{s_0}(B)$ of diameter $\leq \varepsilon(1+|s-s_0|k/c_s)$ and so

$$\alpha(T_{s_0}(B)) \leq \varepsilon(1+|s-s_0|k/c_s).$$

Letting $\varepsilon \downarrow \alpha(T_s(B))$ we obtain

$$c_{s_0}\gamma(B) \leq \alpha(T_{s_0}(B)) \leq \alpha(T_s(B))(1+|s-s_0|k/c_s),$$

whence $c_{s_0} \leq 2c_s(1+|s-s_0|k/c_s)$, that is $c_{s_0} \leq 2c_s + 2k|s-s_0|$, which clearly shows that c_s are upper bounded by a number $c > 0$, as desired.

Next, suppose that for a certain $s < 1$, T_s is surjective (equivalently, bijective). Define $F_t = (T_s - T_t)T_s^{-1} : X \to X$ for $s \leq t \leq 1$. We have $F_t = (t-s)ST_s^{-1}$ whence, by the continuity of S and T_s^{-1}, F_t is continuous. Now, let $k > 0$ be such that S is k-γ-Lipschitz. If $B \subset X$ is bounded, then by (1) we have

$$\gamma(F_t(B)) = (t-s)\gamma(ST_s^{-1}(B)) \leq (t-s)k\,\gamma(T_s^{-1}(B)) \leq (t-s)\frac{k}{c}\cdot\gamma(B).$$

This implies that F_t is γ-condensing whenever

$s \leq t < s + \frac{c}{k}$, $t \leq 1$. Let $y \in X$ be fixed arbitrary and put $G_t = F_t + y$. Clearly, G_t is γ-condensing for $s \leq t < s + \frac{c}{k}$, $t \leq 1$. We will prove that the set of fixed points of the mappings

G_t, $s \le t < s + \frac{e}{k}$, $t \le 1$, is bounded. Indeed, if $x \in \mathrm{Fix}(G_t)$, then $x = F_t x + y$ or equivalently, $x = x - T_t T_s^{-1} x + y$. Hence $T_s^{-1} x = T_t^{-1} y$, whence by (iv), the set

$\{T_s^{-1} x : x \in \mathrm{Fix}(G_t),\ s \le t < s + \frac{e}{k},\ t \le 1\}$ is bounded. Since, by (i) and (ii) T_s maps bounded sets into bounded sets, this implies that $\{x : x \in \mathrm{Fix}(G_t),\ s \le t < s + \frac{e}{k},\ t \le 1\}$ is also bounded, as claimed. Let $r > 0$ be such that $|x| < r$ whenever $x \in \mathrm{Fix}(G_t)$, $s \le t < s + \frac{e}{k}$, $t \le 1$ and put $U = \{x \in X : |x| < r\}$. For each $t \le 1$ satisfying $s \le t < s + \frac{e}{k}$, define

$$H : J \times \bar{U} \to X,\quad H(\lambda, x) = G_{(1-\lambda)s + \lambda t}\, x.$$

It is easy to see that $H(0,.) = G_s \equiv y$ and $H(1,.) = G_t$, as mappings from $\bar{U}$ to X, are homotopic and since $G_s \equiv y$ is essential, because $y \in U$, G_t will be essential too. Thus, there is $x_y \in U$ such that $x_y = G_t x_y$, i.e. $y = T_t T_s^{-1} x_y$. Since y was arbitrary fixed in X, we get that T_t is surjective for any t satisfying $s \le t < s + \frac{e}{k}$, $t \le 1$. Now it is clear that if $T_o = T$ is surjective, then after a finite number of steps, it follows that $T_1 = T - S$ is also surjective.

Remark 4.1. In particular, conditions (i)-(iv) in Theorem 4.2 are satisfied if T and S are Lipschitz and there exists $e > C$ such that $e d(x,y) \le |T_s x - T_s y|$ for all $x, y \in Y$ and $s \in J$. Indeed, in this case, conditions (i)-(iii) clearly hold. To prove (iv), let $x \in X$. We fix $s_o \in J$ such that $T_{s_o}^{-1} x \ne \emptyset$ and we put

$y_s = T_s^{-1}x$ whenever for some $s \in J$ one has $T_s^{-1}x \neq \emptyset$. Since $0 = |T_s y_s - T_{s_0} y_{s_0}| \geq |T_s y_s - T_s y_{s_0}| - |T_s y_{s_0} - T_{s_0} y_{s_0}|$, we get

$$cd(y_s, y_{s_0}) \leq |T_s y_s - T_s y_{s_0}| \leq |T_s y_{s_0} - T_{s_0} y_{s_0}| =$$

$$= |s - s_0| |S y_{s_0}| \leq |S y_{s_0}|,$$

for each $s \in J$ with $T_s^{-1}x \neq \emptyset$. Hence $\{ T_s^{-1}x : s \in J \}$ is bounded.

In this special case Theorem 4.2 reduces to a result of F.E. Browder [2].

Remark 4.2. An other consequence of Theorem 4.2 is the following hyperaccretivity criterion: If X is a real Banach space and $F : X \to X$ is accretive and γ-Lipschitz, then F is hyperaccretive. For the proof it is sufficient to take $T = I$, the identity of X, and $S = -F$ and to apply Theorem 4.2. This result can also be derived from a more general hyperaccretivity criterion due to V. Barbu [1], Corollary 3.3.2.

Remark 4.3. A topological transversality theorem for multivalued γ-condensing mappings T, having Tx closed convex for all x, can be proved similarly. It can be used instead of the degree theory, to prove certain other perturbation theorems as, for instance, some results of J.R.L. Webb [11].

5. Application. We will study the existence and the uniqueness of the solution to the problem

$$u''(t) + p(t)u'(t) = f(t, u(t)) + g(t), \quad t \in J \tag{2}$$

$$u(0) = a, \quad u(1) = b \tag{3}$$

in a real Banach space X, where $p \in C(J; \mathbb{R})$, $g \in C(J; X)$, $f \in C(J \times X; X)$ and $a, b \in X$. We look for solutions in C^2.

Let $C_b^2 = \{u \in C^2 : u(0) = a, \ u(1) = b\}$. Clearly, C_b^2 is a closed convex subset of C^2.

Theorem 5.1. Assume

(a) f is uniformly continuous on $J \times B$ for any bounded $B \subset X$;

(b) There exists $k \geq 0$ such that

$$(4) \qquad \gamma(f(t,B)) \leq k\,\gamma(B), \quad \text{for any bounded } B \subset X;$$

(c) There exists $c > 0$ such that

$$(5) \qquad (f(t,x_1)-f(t,x_2),\ x_1-x_2)_+ \geq c|x_1-x_2|^2$$

for all $x_1, x_2 \in X$ and $t \in J$.

Then (2)-(3) has exactly one solution $u \in C^2$.

Proof. We shall apply Theorem 4.2 with $C = C(J;X)$ instead of X and $Y = C_b^2$ to the mappings $T = L$,
$L : C_b^2 \to C$, $(Lu)(t) = u''(t)+p(t)u'(t)$ and
$S = F$, $F : C_b^2 \to C$, $(Fu)(t) = f(t,u(t))$.

1) We will prove that F is γ-Lipschitz. At the beginning we show that

$$(6) \qquad \gamma(f(J \times B)) \leq k\,\gamma(B) \quad \text{for all bounded } B \subset X.$$

Indeed, for $\varepsilon > 0$ arbitrary fixed, by (a), we have that for each $\bar{t} \in J$ there is a neighbourhood $V(\bar{t},\varepsilon)$ of $\bar{t}$ such that

$$|f(t,x)-f(\bar{t},x)| < \varepsilon \quad \text{for all } t \in V(\bar{t},\varepsilon) \text{ and } x \in B.$$

In consequence

$$\gamma(f(V(\bar{t},\varepsilon) \times B)) \leq \gamma(f(\bar{t},B))+2\varepsilon,$$

whence, taking into account (4) and the compactness of J, we get

$$\gamma(f(J \times B)) \leq k\,\gamma(B)+2\varepsilon$$

Now (6) follows if we take $\varepsilon \to 0$.

The continuity of F follows by the continuity of f. Now let $D \subset C_b^2$ be bounded. From (6), taking $B = D(J)$, we see that $F(D)$ is bounded. On the other hand, since D is bounded in C^2, it follows that D is equicontinuous. Hence, $\alpha(D) = \sup\{\alpha(D(t)) : t \in J\}$. Also, the equicontinuity of D together with the uniform continuity of f, implies that $F(D)$ is equicontinuous too and so

$$\alpha(F(D)) = \sup\left\{ \alpha(F(D)(t)) : t \in J \right\}.$$

Inasmuch as by (4),

$$\alpha(F(D)(t)) = \alpha(f(t,D(t))) \leq 2\,\gamma(f(t,D(t))) \leq$$

$$\leq 2k\,\gamma(D(t)) \leq 2k\,\alpha(D(t)) \leq 2k\,\alpha(D)$$

we deduce that

$$\alpha(F(D)) \leq 2k\,\alpha(D), \tag{7}$$

whence all the more

$$\gamma(F(D)) \leq 4k\,\gamma(D). \text{ Finally, since } \gamma(D) \leq \gamma_2(D),$$

$$\gamma(F(D)) \leq 4k\,\gamma_2(D) \quad \text{for all bounded } D \subset C_b^2, \tag{8}$$

which shows that F is γ-Lipschitz.

2) Obviously L maps bounded subsets of C_b^2 into bounded sets of C.

3) $T_s = L - sF\,(s \in J)$ is injective. This being clear for $s = 0$ we may assume $s > 0$. Let $T_s u = T_s v$ where $u, v \in C_b^2$. Let $t_o \in J$ such that $\|u(t_o) - v(t_o)\| = \max\{\|u(t) - v(t)\| : t \in J\}$. We may suppose that $t_o \in]0,1[$ because $u(0) - v(0) = u(1) - v(1) = 0$. Then $(x^*, u'(t_o) - v'(t_o)) = 0$ and $(x^*, u''(t_o) - v''(t_o)) \leq 0$ for every $x^* \in \mathcal{F}(u(t_o) - v(t_o))$. Consequently, since

$u''(t) - v''(t) + p(t)(u'(t) - v'(t)) = s(f(t,u(t)) - f(t,v(t)))$, we get

$0 \geqslant (f(t_0,u(t_0))-f(t_0,v(t_0)),u(t_0)-v(t_0))_+$, whence, by (5) $u(t_0)-v(t_0) = 0$, that is $u = v$.

4) The following step is to prove that for each $s \in J$ there exists $c_s > 0$ such that

(9) $$c_s\,\gamma_2(T_s^{-1}(B)) \leqslant \gamma(B) \quad \text{for any bounded } B \subset C.$$

Such an inequality clearly holds for $s = 0$ because $T_0 = L : C_b^2 \to C$ is linear bounded and bijective. Hence we may assume $s > 0$. Let $B \subset C$ be bounded and let $\varepsilon > \alpha(B)$ be arbitraryly fixed. Then B admits a finite cover by subsets B_j of diameter $\leqslant \varepsilon$. We have only to prove that each set $T_s^{-1}(B_j)$ can be convered by finitely many subsets $B_{j\ell}$ of $\mathrm{diam}_2 B_{j\ell} \leqslant \varepsilon/(2\tilde{c}_s)$, $\tilde{c}_s$ being a constant independent of B, ε, j and ℓ (we have denoted by diam_2 the diameter with respect to the norm $\|\cdot\|_2$):

4a) Let $g_i \in B_j$ and $u_i = T_s^{-1} g_i$, $i=1,2$ with $g_1 \neq g_2$. Consider $t_0 \in]0,1[$ such that $\|u_1-u_2\| = |u_1(t_0)-u_2(t_0)|$ (>0). From

(10) $$u_1''(t)-u_2''(t)+p(t)(u_1'(t)-u_2'(t)) = s(f(t,u_1(t))-f(t,u_2(t)))+ \\ +g_1(t)-g_2(t)$$

we deduce

$$s(x^*,f(t_0,u_1(t_0))-f(t_0,u_2(t_0)))+(x^*,g_1(t_0)-g_2(t_0)) \leqslant 0$$

for all $x^* \in \mathcal{F}(u_1(t_0)-u_2(t_0))$. Since $\|g_1-g_2\| \leqslant \varepsilon$, by (5) this yields

(11) $$\|u_1-u_2\| \leqslant \varepsilon/(sc).$$

4b) Applying (7) to $D = T_s^{-1}(B_j)$ and taking into account (11) we may infer that $T_s^{-1}(B_j)$ can be represented as an union of finitely many subsets $B_{j\ell}$ such that diam $F(B_{j\ell}) \leq 2k\varepsilon/(sc)+\varepsilon$. In the following we shall fix our attention to some subset $B_{j\ell}$. Suppose that $u_1, u_2 \in B_{j\ell}$.

Then $|f(t,u_1(t))-f(t,u_2(t))| \leq 2k\varepsilon/(sc)+\varepsilon$ for *all* $t \in J$ and by (10) and (11) we get

$$\|u_1''-u_2''\| \leq M_p\|u_1'-u_2'\|+ 2k\varepsilon/c+2\varepsilon, \tag{12}$$

where $M_p = \max\{|p(t)| : t \in J\}$.

4c) u_1 and u_2 being as in 4b) denote $v = u_1-u_2$ and $q=\|v'\| = |v'(t_o)|$, where $t_o \in J$. Let μ be such that $|\mu| \leq 1/2$ and $t_o+\mu \in J$ and put $\delta = |\mu|$. Using Taylor's formula

$$v(t_o+\mu) = v(t_o)+\mu v'(t_o) + \frac{\mu^2}{2} v''(t_o+\theta\mu) \quad \text{for some } \theta \in J,$$

by (11) and (12) one deduces

$$\delta q \leq 2\varepsilon/(sc) + \phi(q)\delta^2/2, \tag{13}$$

where $\phi(q) = M_p q+2k\varepsilon/c+2\varepsilon$. It is easy to see that

$\xi^2/\phi(\xi) > 4\varepsilon/(sc)$ for all $\xi > \varepsilon Q$, where Q is some nonnegative constant independent of ε. Assume $q > \varepsilon Q$. Then $\phi(q) < q^2sc/(4\varepsilon)$ and by (13) one has

$$q < \frac{2\varepsilon}{sc}\cdot\frac{1}{\delta} + \frac{q^2sc}{8\varepsilon}\,\delta \tag{14}$$

This implies that $4\varepsilon/(scq) \geq 1/2$ or equivalently, $q \leq 8\varepsilon/(sc)$. Indeed, if $4\varepsilon/(scq) < 1/2$, choosing $\delta = 4\varepsilon/(scq)$ in (14), one obtains $q < q/2+q/2$, a contradiction. Thus, if $q > \varepsilon Q$ then $q \leq 8\varepsilon/(sc)$. Therefore,

$$\|u_1'-u_2'\| \leq \max\{\varepsilon Q,\ 8\varepsilon/(sc)\} \tag{15}$$

By (11), (12) and (15), $\operatorname{diam}_2 B_{j\ell} \leq \varepsilon/(2c_s)$, where $1/(2c_s) = \max\{Q, 8/(sc), M_p \max\{Q, 8/(sc)\} + 4k/c+2\}$, as desired.

5) For each $s \in J$, T_s^{-1} is continuous. To show this, let $g_n \xrightarrow{C} g^*$ as $n \to \infty$ and $u_n = T_s^{-1} g_n$. Since $\gamma(\{g_n : n \geq 1\}) = 0$, by (9) the sequence (u_n) contains a subsequence which converges is C^2 to some u^*. By $g_n = T_s u_n$ and the continuity of T_s, we obtain $g^* = T_s u^*$, i.e. $u^* = T_s^{-1} g^*$, whence it follows that even the entire sequence (u_n) converges to $T_s^{-1} g^*$.

6) Condition (iv) in Theorem 4.2 is satisfied if for each $g \in C$ the set of all solutions $u \in C_b^2$ to

$$u'' + p(t)u' = sf(t,u) + g(t) \tag{16}$$

for $s \in J$, is bounded in C^2. Let $u_s \in C_b^2$ a solution to (16) and $u_0 \in C_b^2$ the solution to (16) in case $s = 0$. Denote $v_s = u_s - u_0$. Then

$$v_s'' + p(t)v_s' = sf(t, u_0(t) + v_s) .$$

Thus, wee have to prove the a priori boundedness of the set of all solutions $v \in C^2$ satisfying

$$v'' + p(t)v' = sf(t, u_0(t) + v) \tag{17}$$

and $v(0) = v(1) = 0$, for $s \in J$. But this follows by Lemmas 4 and 5 in [8] because, by (5) we have

$$(f(t,u_0(t)+x),x)_+ \geq -(-f(t,u_0(t)),x)_+ + c|x|^2 \geq$$

$$\geq c|x|^2 - \|f(.,u_0(.))\|\,|x| > 0,$$

for $|x| > \|f(.,u_0(.))\|/c$, which means that condition (iii) of

Thus, all the assumptions of Theorem 4.2 are satisfied. Therefore, since L is surjective, L-F is surjective too (even bijective), which shows that (2),(3) has exactly one solution $u \in C^2$ for each $g \in C$.

Remark 5.1. The existence of solutions to (2),(3) also follows from our paper [8], by using directly the topological transversality theorem and the a priori bounds technique, but under the additional assumption that k in (4) be sufficiently small. Thus, the advantage of using the perturbation Theorem 4.2 in case of equation (2), consists in the fact that k in (4) may be arbitrar.

Remark 5.2. Conditions (a) and (b) in Theorem 5.1 are, in particular, satisfied if f is completely continuous or if $f = f_1+f_2$ where f_1 is completely continuous and f_2 is Lipschitz.

Remark 5.3. In particular, if $X = \mathbb{R}^n$ the assumptions of Theorem 5.1 reduce to: $f : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous and

$$(f(t,x_1)-f(t,x_2),x_1-x_2) \geqslant c|x_1-x_2|^2 \quad \text{for all} \quad x_1,x_2 \in \mathbb{R}^n, t \in J.$$

In this case the existence and the uniqueness of solution to (2), (3) follows from [5], Theorem II 3.3, Theorem V 2.2.

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