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SOME REMARKS ON CLARKE GENERALIZED GRADIENT OF QUASICONVEX FUNCTIONS

by

RADU PRECUP

(Cluj-Napoca)

1. Let X be a Banach space, X^* its dual and $f:D \longrightarrow R$, $D \subset X$, a function. H.J.Greenberg and W.P.Pierskalla [2] defined the <u>quasisubdifferential</u> of f at $x \in D$, by

(1) $$\partial f^*(x) = \left\{ x^* \in X^*;\ (x^*, y-x) \geqslant 0,\ \ y \in D \ \text{ implie } \ f(y) \geqslant f(x) \right\}.$$

This notion is as closely related to quasiconvex functions as the usual subdifferential is to convex functions.

Assume that f is defined and locally Lipschitz (i.e. Lipschitz on each bounded subset) in a neighbourhood of x. The <u>Clarke generalized gradient</u> of f at x is defined by

(2) $$\partial f(x) = \left\{ x^* \in X^*; (x^*, v) \leqslant f^{o}(x;v) \ \text{ for all } \ v \in X \right\},$$

where $f^{o}(x;v)$ is the <u>generalized directional derivative</u> at x in the direction v, namely

(3) $$f^{o}(x;v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}(f(y+tv) - f(y)).$$

Recall that

$$f^{o}(x;v) = \max \left\{ (x^*, v);\ \ x^* \in \partial f(x) \right\}.$$

In this note some relations between ∂f and ∂f^* in case of quasiconvex functions f, are established.

2. The function f is said to be <u>quasiconvex</u> if D is convex and

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

for all $x, y \in D$ and $t \in [0,1]$.

LEMMA. <u>Let</u> $f: D \longrightarrow R$ <u>be any function with convex domain</u> $D \subset X$. <u>If</u> $\partial f^{*}(x) \neq \emptyset$ <u>for all</u> $x \in D$, <u>then</u> f <u>is quasiconvex</u>.

<u>Proof</u>. Let $x, y \in D$, $t \in]0,1[$ and $x_t = x + t(y-x)$. Let $x_t^{*} \in \partial f^{*}(x_t)$. Since $x - x_t = -t(y-x)$ and $y - x_t = (1-t)(y-x)$, we have

$$(x_t^{*}, x - x_t) \geq 0 \quad \text{or} \quad (x_t^{*}, y - x_t) \geq 0,$$

whence $f(x_t) \leq \max\{f(x), f(y)\}$. Therefore, f is quasiconvex.

In the following we shall assume that D is a nonempty open convex subset of X and that $f: D \longrightarrow R$ is locally Lipschitz on D.

THEOREM 1. <u>Suppose that</u> $\partial f(x) \neq \{0\}$ <u>for every</u> $x \in D$. <u>The following statements are equivalent:</u>

1°. f <u>is quasiconvex</u>.

2°. $\partial f(x) \setminus \{0\} \subset \partial f^{*}(x)$ <u>for all</u> $x \in D$.

<u>Proof</u>. $1^{\circ} \Rightarrow 2^{\circ}$: Let $x^{*} \in \partial f(x) \setminus \{0\}$ and let $y \in D$ such that $(x^{*}, y-x) \geq 0$. Since $x^{*} \neq 0$, in each neighbourhood of y there is a point y_1 such taht $(x^{*}, y_1 - x) > 0$. From

$$0 < (x^{*}, y_1 - x) \leq f^{\circ}(x; y_1 - x),$$

it follows that there are two sequences $z_n \rightarrow x$ and $t_n \downarrow 0$ such that

$$0 < f(z_n + t_n(y_1 - x)) - f(z_n).$$

On the other hand, the quasiconvexity of f implies that

$$f(z_n + t_n(y_1 - x)) \leq f(z_n + y_1 - x).$$

Hence, $f(z_n) < f(z_n + y_1 - x)$ and letting $n \rightarrow \infty$ we obtain $f(x) \leq f(y_1)$. Consequently, $f(x) \leq f(y)$, which shows that $x^{*} \in \partial f^{*}(x)$.

The implication $2^\circ \Rightarrow 1^\circ$ follows by Lemma.

THEOREM 2. *Let* $\partial f(x) \not\ni 0$ *for any* $x \in D$. *The following statements are equivalent:*

1°. f *is quasiconvex.*

2°. $f^\circ(x;y-x) \geq 0$ *implies* $f(y) \geq f(x)$

$f^\circ(x;x-y) < 0$ *implies* $f(y) > f(x)$ $(x,y \in D)$.

3°. $f^\circ(x;y-x) \geq 0$ *implies* $f(y) \geq f(x)$ $(x,y \in D)$.

4°. $\partial f(x) \subset \partial f^*(x)$ *for each* $x \in D$.

Proof. $1^\circ \Longrightarrow 2^\circ$: Suppose that f is quasiconvex and $f^\circ(x;y-x) \geq 0$, $y \in D$. Then, there exists $x^* \in \partial f(x)$, clearly $x^* \neq 0$, such that $(x^*,y-x) \geq 0$, and the inequality $f(y) \geq f(x)$ follows as in the proof on Theorem 1.

Assume now that $f^\circ(x;x-y) < 0$ and $y \in D$. Then, it is easy to see that $f^\circ(x;y-x) > 0$. In consequence, $f(x_t) \geq f(x)$ for all $t \in [0,1]$, where $x_t = x + t(y-x)$. Suppose, nevertheless that $f(y) = f(x)$. Then, the quasiconvexity of f implies $f(x_t) \leq f(x)$ for all $t \in [0,1]$. Hence, $f(x_t) = f(x)$ for all $t \in [0,1]$. Since $f^\circ(x;x-y) < 0$ and the map $\partial f: D \rightarrow 2^{X^*}$ is weak*-upper-semicontinuous, there is an open neighboarhood V_o of x included in D such that $f^\circ(x';x'-y) < 0$ and consequently $f(y) = f(x) \geq f(x')$, for all $x' \in V_o$. On the other hand by $f^\circ(x;x-x_t) < 0 (t \in]0,1])$, there exists an open neighbourhood V_t of x_t such that $f^\circ(x;z-x) > 0$ and consequently $f(z) \geq f(x)$, for all $z \in V_t$. The sets V_o and V_t, $t \in [0,1]$, represent an open cover of the compact segment $[x,y]$. Let V_o, V_{t_i}, $i=1,2,\ldots,n$, $0 < t_1 < t_2 < \ldots < t_n \leq 1$, be a finite subcover of $[x,y]$. We have $f(x') = f(x)$ for all $x' \in V_o \cap V_{t_1}$. This implies that $\partial f(x') = \{0\}$ for any $x' \in V_o \cap V_{t_1}$, a contradiction. Therefore, $f(y) > f(x)$.

The implication $2^o \Rightarrow 3^o$ is obvious.

$3^o \Rightarrow 4^o$: Let $x^* \in \partial f(x)$. Suppose that $(x^*, y-x) \geq 0$, $y \in D$. Then, $f^o(x;y-x) \geq 0$ and so $f(y) \geq f(x)$. Hence $x^* \in \partial f^*(x)$.

The implication $4^o \Rightarrow 1^o$ follows by Lemma.

Remark. For any function f satisfying 2^o in Theorem 2, one has

$$\min \left\{ f^o(x;y-x), \ - f^o(y;y-x) \right\} \leq 0 \tag{4}$$

for all $x, y \in D$.

In the particular case when f is Gâteaux differentiable on D, there is known that $\partial f(x) = \{ \nabla f(x) \}$ and $f^o(x;v) = (\nabla f(x), v)$ and so, 3^o in Theorem 2 becomes just the condition of pseudoconvexity of f, while (4) expresses the pseudomonotonicity of ∇f, both in the sense of S.Karamardian [3] .

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Faculty of Mathematics
University of Cluj
34oo-Cluj-Napoca