R. PRECUP

Topological transversality and boundary problems for second-order functional differential equations

1. Introduction

This article deals with boundary value problems

(P1) $u''(t) - u(t) \in Au(t) + (Ku)(t) + f(t, u(t))$, a.e. on $]0, 1[$,

$u'(0) \in b_1(u(0))$, $u'(1) \in -b_2(u(1))$;

and

(P2) $u''(t) \in Au(t) + (Ku)(t) + g(t, u(t), u'(t))$, a.e. on $]0, 1[$,

$u'(0) \in b_1(u(0))$, $u(1) = 0$,

in a real Hilbert space E, where A, b_1 and b_2 are maximal monotone subsets of $E \times E$, K is a completely continuous operator and f and g satisfy the Carathéodory conditions and additional Lipschitz inequalities.

The main tool that we use to establish the existence of solutions is the topological transversality theorem (the Leray–Schauder principle) for condensing operators (see the author's paper [6] or W. Krawcewicz [4]). For completely continuous operators this theorem is due to A. Granas [2]. In Section 2 we state a much more general version of the topological transversality theorem which was obtained by the author [8].

In the absence of nonmonotone terms, i.e., when K, f and g are null, Problems (P1) and (P2) were studied by V. Barbu [1] and N. Pavel [5] in a more general frame.

2. Generalized topological transversality

Let X be a normal topological space, let M be a proper closed subset of X, Y and let N be a proper subset of Y. Consider a class of operators

$$\mathcal{A} \subset \{G : X \to Y;\ G^{-1}(N) \cap M = \emptyset\}.$$

$G_1 \in \mathcal{A}$, such that the restrictions of G and G_1 to M coincide, i.e., $G/_M = G_1/_M$, one has $G_1^{-1}(N) \neq \emptyset$. Otherwise, G is said to be inessential.

Let $\sim$ be an equivalence relation on $\mathcal{A}$ such that

$$\text{(A)} \qquad G/_M = G_1/_M \text{ implies } G \sim G_1.$$

We are interested in the case when the equivalence classes contain either only essential operators or only inessential operators. The next condition is sufficient to have such a case.

(H) If $G \sim G_1$, then there is $H : [0, 1] \times X \to Y$ such that $H(0, \cdot) = G$, $H(1, \cdot) = G_1$,

$$\mathrm{cl}(\bigcup \{H(\lambda, \cdot)^{-1}(N);\ \lambda \in [0, 1]\}) \cap M = \emptyset,$$

and $H(\theta(\cdot), \cdot) \in \mathcal{A}$ for every continuous function $\theta : X \to [0, 1]$ satisfying $\theta(x) = 1$ for all $x \in M$.

Theorem 1. Assume that hypotheses (A) and (H) hold. Let G and G_1 be in $\mathcal{A}$ such that $G \sim G_1$. Then G and G_1 are both essential or both inessential.

For the proof and some applications, see [8].

We shall use Theorem 1 in the particular case: $Y = B \times B$, B being a real Banach space, $X = \mathrm{cl}(U)$, where U is a nonvoid bounded open subset of B, M is the boundary ∂U of U, $N = \{(x, x);\ x \in B\}$ and $\mathcal{A}$ is the class of all mappings $G : \mathrm{cl}(U) \to B \times B$ of the form

$$Gx = (x, Tx),\ x \in \mathrm{cl}(U),$$

where $T : \mathrm{cl}(U) \to B$ is a condensing operator, fixed point free on ∂U. Also, define $G \sim G_1$ if there is $h : [0, 1] \times \mathrm{cl}(U) \to B$ condensing, such that $h(0, \cdot) = T$, $h(1, \cdot) = T_1$ and $h(\lambda, x) \neq x$ for all $\lambda \in [0, 1]$ and $x \in \partial U$, where $Gx = (x, Tx)$ and $G_1x = (x, T_1x)$. In this case, $H(\lambda, x) = (x, h(\lambda, x))$.

Corollary 1. Let $T : \mathrm{cl}(U) \to B$ be a condensing operator and $x_0 \in U$. If $\lambda(Tx - x_0) \neq x - x_0$ for all $x \in \partial U$ and $\lambda \in [0, 1]$, then T has at least one fixed point.

For the proof it is sufficient to see that $Gx = (x, Tx) \sim G_1x = (x, x_0)$ and that G_1 is essential (see [8]).

3. The existence results

Let E be a real Hilbert space whose norm and inner product are denoted by $|\ |$ and $\langle\ ,\ \rangle$; A, b_1 and b_2 maximal monotone subsets of $E \times E$ with $(0, 0) \in A \cap b_1 \cap b_2$. For $\lambda > 0$, let $A_\lambda x = \lambda^{-1}(x - (I + \lambda A)^{-1}x)$. As in [5] we require that

$$\langle A_\lambda x, y \rangle \geq 0 \text{ for all } (x, y) \in b_i \quad (i = 1,2). \tag{3.1}$$

Theorem 2. Let the assumption (3.1) hold for $i = 1,2$. Suppose that K is a completely continuous operator from $L^2(0, 1; E)$ into itself so that

$$\limsup \|Ku\| / \|u\| \leq c_0 \text{ as } \|u\| \to \infty, \tag{3.2}$$

and $f : [0, 1] \times E \to E$ is a function such that

$$f(\cdot, x) \text{ is measurable for all fixed } x \in E; \tag{3.3}$$

$$|f(t, x_1) - f(t, x_2)| \leq c_1 |x_1 - x_2| \text{ for all } x_1, x_2 \in E,\ t \in [0, 1], \tag{3.4}$$

where $c_0 + c_1 < 1$;

$$f(\cdot, 0) \in L^2(0, 1; E). \tag{3.5}$$

Then there is at least a solution $u \in H^2(0, 1; E)$ of Problem (P1).

Proof. Let D and $\bar{A}$ be the subsets of $L^2(0, 1; E) \times L^2(0, 1; E)$:

$$D = \{(u, -u''); u \in H^2(0, 1; E),\ u'(0) \in b_1(u(0)), u'(1) \in -b_2(u(1))\},$$

$$\bar{A} = \{(u, v); v(t) \in \mathrm{A}u(t), \text{ a.e. on } [0, 1]\}.$$

It is known (see [5, pp. 206]) that $D + \bar{A}$ is maximal monotone and that $(I + D + \bar{A})^{-1}$ is nonexpansive and $L^2(0, 1; E)$.

Also, consider the operator $F : L^2(0, 1; E) \to L^2(0, 1; E)$,

$$(Fu)(t) = -f(t, u(t)),\ u \in L^2(0, 1; E),\ t \in [0, 1].$$

Conditions (3.3)–(3.5) imply that F is well-defined. Moreover, according to (3.4), F is a c_1-contraction and hence $F - K$ is a c_1-set-contraction. Thus, $T = (I + D + \bar{A})^{-1}(F - K)$ is condensing. In addition, one has

$$\ldots \le \|(F-K)u\| \le \|F(0)\| + \|Fu - F(0)\| + \|Ku\|$$

$$\le \|F(0)\| + \|Ku\| + c_1 \|u\|.$$

Whence, by (3.2) and $c_0 + c_1 < 1$, we deduce that there is $R > 0$ such that $\|Tu\| < \|u\|$ for $\|u\| \ge R$. Thus, we may apply Corollary 1 with $B = L^2(0, 1; E)$, $U = \{u \in L^2(0, 1; E);\ \|u\| < R\}$ and $x_0 = 0$. It follows that T has a fixed point, i.e., Problem (P1) has a solution.

Theorem 3. Let the assumption (3.1) hold for $i = 1$. Suppose that K is a completely continuous operator from $H^1(0, 1; E)$ into $L^2(0, 1; E)$ such that

$$\limsup \|Ku\| / \|u\|_1 \le c_0 \text{ as } \|u\|_1 \to \infty \tag{3.6}$$

and let $g : [0, 1] \times E^2 \to E$ satisfy

$$g(\cdot, x, y) \text{ is measurable for all fixed } x, y \in E; \tag{3.7}$$

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le c_1 |x_1 - x_2| + c_2 |y_1 - y_2| \tag{3.8}$$

for all $x_1, x_2, y_1, y_2 \in E$ and $t \in [0, 1]$, where

$$4\pi^{-2}(c_1 + 1) + 2\pi^{-1}(c_0 + c_2) < 1; \tag{3.9}$$

$$g(\cdot, 0, 0) \in L^2(0, 1; E). \tag{3.10}$$

Then there is at lest a solution $u \in H^2(0, 1; E)$ of Problem (P2).

Proof. Let $b_2 = \{0\} \times E$, $B = \{u \in H^1(0, 1; E);\ u(1) = 0\}$ and let $F : B \to L^2(0, 1; E)$,

$$(Fu)(t) = u(t) - g(t, u(t), u'(t)),\ u \in B,\ t \in [0, 1].$$

For $u \in B$, it is true the Wirtinger inequality is (see [3])

$$\|u\| \le 2\pi^{-1} \|u'\|. \tag{3.11}$$

By (3.8) and (3.11) it follows that F is a $(2\pi^{-1}(c_1 + 1) + c_2)$- Lipschitzian operator from $B \subset H^1(0, 1; E)$ into $L^2(0, 1; E)$. Thus, $F - K$ is $(2\pi^{-1}(c_1 + 1) + c_2)$set-Lipschitzian. Next we show that $(I + D + \bar{A})^{-1}$ is $2\pi^{-1}$-Lipschitzian as an operator from $L^2(0, 1; E)$ into $H^1(0, 1; E)$. In fact, let $v_i = (I + D + \bar{A})^{-1} u_i$, $i = 1,2$, and let $u = u_1 - u_2$ and $v = v_1 - v_2$. We have

$$v''(t) \in v(t) - u(t) + Av_1(t) - Av_2(t), \text{ a.e. on } [0, 1].$$

Since

$$|v'(t)|^2 = \langle v'(t), v(t)\rangle' - \langle v''(t), v(t)\rangle,$$

this yields

$$|v'(t)|^2 = \langle v'(t), v(t)\rangle' - |v(t)|^2 + \langle u(t), v(t)\rangle$$

$$- \langle Av_1(t) - Av_2(t), v(t)\rangle \le \langle v'(t), v(t)\rangle' + \langle u(t), v(t)\rangle.$$

It follows that

$$\| v' \|^2 \le \| u \| \| v \| + \langle v'(1), v(1)\rangle - \langle v'(0), v(0)\rangle$$

$$\le \| u \| \| v \| \le 2\pi^{-1} \| u \| \| v' \| .$$

Thus, $\| v \|_1 = \max \{ \| v \| , \| v' \| \} \le 2\pi^{-1} \| u \|$, as claimed. Therefore, $T = (I + D + \bar{A}^{-1}(F - K)$ is a set-contraction from B into B with respect to the norm of $H^1(0, 1; E)$.

In addition, using (3.6) and (3.9), it is easily seen that $R > 0$ such that

$$\| Tu \|_1 < \| u \|_1 \text{ for } u \in B, \ \| u \|_1 \ge R.$$

Thus, we may apply, once again, Corollary 1.

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