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POSITIVE SOLUTIONS OF THE INITIAL VALUE PROBLEM FOR AN INTEGRAL EQUATION MODELING INFECTIOUS DISEASE

by

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1.INTRODUCTION

We deal with the nonlinear integral equation

(1) $$x(t)=\int_{t-\tau}^{t} f(s,x(s))ds$$

which is a model for the spread of certain infectious diseases with a contact rate that varies seasonally.In this equation $x(t)$ is the proportion of infectives in a population at time t , τ is the lenght of time an individual remains infectious and $f(t,x(t))$ represents the proportion of new infectives per unit time.

In the papers (1),(3),(5), (6) can be found sufficient conditions for the existence of nontrivial periodic nonnegative and continuous solutions to equation (1), in the case of a periodic contact rate ($f(t+\omega,x)=f(t,x)$, $f(t,0)=0$).

In this paper we deal with positive and continuous solutions $x(t)$ of equation (1), for $-\tau\le t\le T$, when we know the proportion $\phi(t)$ of infectives in the population for $-\tau\le t\le 0$, i.e.

(2) $$x(t)=\phi(t) \quad , \text{ for } -\tau\le t\le 0 .$$

Clearly, we must supose that ϕ satisfies the condition

$$b = \phi(0) = \int_{-\tau}^{0} f(s,\phi(s))\, ds .$$

this condition problem (1)-(2) is equivalent with the initial problem:

$$x'(t) = f(t,x(t)) - f(t-\tau, x(t-\tau)), \quad 0 \le t \le T$$

$$x(t) = \phi(t), \quad -\tau \le t \le 0 .$$

2. MAIN RESULTS

Let us list our assumptions:

(i) $f(t,x)$ is nonnegative and continous for $-\tau \le t \le T$ and

(ii) $\phi(t)$ is continous and $0 < a \le \phi(t)$ for $-\tau \le t \le 0$ satisfies condition (3).

(iii) there exists an integrable funtion $g(t)$ such that $f(t,x) \ge g(t)$ for $-\tau \le t \le T$ and $x \ge a$

$$\int_{t-\tau}^{t} g(s)\, ds \ge a \qquad \text{for } 0 \le t \le T .$$

(iv) there exists a positive function $h(x)$ such that $1/h(x)$ locally integrable on $(a, +\infty)$,

$$f(t,x) \le h(x) \qquad \text{for } 0 \le t \le T \text{ and } x \ge a$$

$$T < \int_{b}^{\infty} (1/h(x))\, dx .$$

Theorem 1. *Suppose that assumptions (i)-(iv) are satisfied. equation (1) has at least one continuous solution $x(t)$, $\ge a$, for $-\tau \le t \le T$, which satisfies condition (2).*

Proof. We shall use the topological transversality theorem (2)) and the technique of a priori bounds on solutions.

Let $\mathcal{E}$ be the Banach space of all continuous functions $x(t)$, $t \le T$, with norm

$$\|x\| = \max_{0 \le t \le T} |x(t)| .$$

Consider the following closed subset of $\mathcal{E}$:

$$X = \{ x \in \mathcal{E} \;;\; x(0) = b \text{ and } x(t) \ge a \text{ for } 0 \le t \le T \} .$$

For each $\lambda \in [0,1]$ we define the operator

$$H_\lambda : X \to \mathcal{E},$$

$$H_\lambda x(t) = (1-\lambda) b + \lambda \int_{t-\tau}^{t} f(s,x(s))\, ds , \quad 0 \le t \le T ,$$

where $f(s,x(s)) = f(s,\phi(s))$ for $-\tau \le s < 0$.

It is easy to show, by Theorem of Ascoli-Arzelá, that the operators H_λ are completely continuous from X to $\mathcal{E}$. In addition, by (3),(iii) and $b \ge a$, we have for each $\lambda \in [0,1]$

$$H_\lambda(X) \subset X .$$

Now we shall establish a priori bound for the solutions to the problems:

$$(1)_\lambda \qquad x(t) = H_\lambda x(t), \quad 0 \le t \le T$$

$$x(t) = \phi(t), \quad -\tau \le t \le 0 .$$

Let $x(t)$ be a solution to $(1)_\lambda$. Then, for each $t \in [0,T]$, we have

$$x'(t) = \lambda f(t,x(t)) - \lambda f(t-\tau, x(t-\tau))$$

and since $f(t,x)$ is nonnegative

$$x'(t) \le \lambda f(t,x(t)) .$$

Now, since $x(t) \ge a$ for $0 \le t \le T$ and by (7)

$$x'(t) \le \lambda h(x(t)), \quad 0 \le t \le T .$$

This yields

$$\int_{0}^{t} (x'(s)/h(x(s)))\, ds \le \lambda t \le \lambda T \le T$$

for each $0 \le t \le T$. Hence

$$\int_{b}^{x(t)} (1/h(u))\, du \le T, \quad 0 \le t \le T ,$$

which, from (8), implies that there exists $R > b$, independent of λ, such that $x(t) < R$ for all $t \in [0,T]$. Thus, R is a bound for all solutions to $(1)_\lambda$, as claimed.

Therefore, H_λ defines an admissible compact homotopy on the closure of

$$B = \{ x \in X \,;\, x(t) < R \text{ for } 0 \le t \le T \},$$

h is fixed point free on its boundary $\partial B = \{x(t) \in X;\ \|x\| = R\}$. $H_0 \equiv \ell$ is essential (see [2, Theorem 2.2]) and therefore, by topological transversality theorem ([2, Theorem 2.5]), H_1 ssential too. This implies that H_1 has at least one fixed point $\in B$. Clearly,

$$a \le x(t) < R \text{ for } 0 \le t \le T$$

$$x(0) = \ell .$$

proof is complete.

Remark. Let us assume that instead of (iv) one has

(iv') $\limsup_{x \to +\infty} f(t,x)/x = a(t)$ uniformly with respect to $[0,T]$ and $\mu = \sup_{0 \le t \le T} a(t) < +\infty$.

Then, choosing $\alpha > \mu$, there is $\beta \ge 0$ such that

$$f(t,x) \le \alpha x + \beta \quad , \ 0 \le t \le T \ , \ x \ge a$$

e (iv) is satisfied with $h(x) = \alpha x + \beta$ and

$$\int_{\ell}^{+\infty} (1/h(u))\, du = +\infty .$$

n addition in (iv'), $\mu < 1/\tau$ (this is assumtion (H_5) in [3]), taking $\mu < \alpha < 1/\tau$, we can choose $R > \ell$ such that

$$\alpha R + \beta \le (1/\tau) R$$

order that the operator H_1 maps $\bar{B}$ into $\bar{B}$ and so, in this case, orem 1 follows directly from Schauder's fixed point theorem. shall prove that this is also true for an arbitrar value of μ .

Corollary 1. *Suppose that assumptions (i)-(iii) and (iv') are isfied. Then equation (1) has at least one continuous solution), $x(t) \ge a$, for $-\tau \le t \le T$, which satisfies condition (2).*

Proof. We describe a direct proof for this corollary. For s we show that, even in this case, Schauder's fixed point the- m is applicable to H_1 . To do this we use an equivalent norm

on E :

$$\|x\|_\theta = \max_{0 \le t \le T} |x(t)| e^{-\theta t}$$

with a suitable nonnegative number θ .

From (9) we have for each $t \in [0,T]$

$$H_1 x(t) \le \tau\gamma + \int_0^t (\alpha x(s) + \beta)\, ds =$$

$$= \tau\gamma + \beta t + \alpha \int_0^t x(s) e^{-\theta s} e^{\theta s} ds \le \tau\gamma + \beta T + \alpha \| x \|_\theta \int_0^t e^{\theta s}\, ds$$

$$\le \tau\gamma + \beta T + (\alpha/\theta) \| x \|_\theta e^{\theta t}$$

where $\gamma = \max_{-\tau \le t \le 0} f(t, \phi(t))$. Hence

$$H_1 x(t) e^{-\theta t} \le (\alpha/\theta) \| x \|_\theta + \beta T + \tau\gamma .$$

So

$$\| H_1 x \|_\theta \le (\alpha/\theta) \| x \|_\theta + \beta T + \tau\gamma .$$

Now, if we choose $\theta > \alpha$ and $R > \ell$ such that

$$(\alpha/\theta) R + \beta T + \tau\gamma \le R ,$$

we see that H_1 maps

$$V = \{ x \in X \,;\, \|x\|_\theta \le R \}$$

into itself and hence Schauder's fixed point theorem applies to H_1 and V . The proof is thus complete.

Finally, let us consider instead of (iv) a more restrictive condition than (iv'):

(iv'') there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x - y|$$

for all $t \in [-\tau, T]$ and $x , y \in [a , +\infty)$.

Theorem 2. *Suppose that assumptions (i)-(iii) and (iv'') are satisfied. Then equation (1) has an unique continuous solution $x(t)$, $x(t) \ge a$, for $-\tau \le t \le T$, which satisfies condition (2); moreover ,*

$$\max_{0 \le t \le T} |x_n(t) - x(t)| \to 0 \quad as \ n \to +\infty ,$$

where $x_n(t) = \phi(t)$ for $-\tau \le t \le 0$ ($n = 0,1,2,\ldots$) ,

$x_0(t) = \ell$ and $x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s))\, ds$, $0 \le t \le T$ ($n = 1,2,\ldots$).

Proof. Similar arguments as in the proof of Corollary 1 ield to the conclusion that the operator H_1 from X into X is contraction with respect to a suitable norm $\|\cdot\|_\theta$. Thus , anach's fixed point theorem applies.

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This paper is in a final form and no version of it will be ubmitted for publication elsewhere.