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ON THE CONSTRUCTION OF APPROXIMATING LINEAR POSITIVE OPERATORS BY PROBABILISTIC METHODS

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REZUMAT. - Construcţia unor operatori liniari de aproximare prin metode probabilistice. Lucrarea prezintă o metodă probabilistică de construire a unui şir de operatori liniari pozitivi utilizaţi în teoria aproximării uniforme a funcţilor continue de două variabile. Se studiază convergenţa şirului şi se evaluează ordinul de aproximare. În final sunt prezentate exemple care extind în plan rezultatele obţinute în [2].

1. Introduction. Connections between probability and positive linear operators are discussed in many papers. Also a lot of generalizations and investigations of the classical operators of discret type (Bernstein, Szasz, Mirakyan, Meyer-König and Zeller and others) were studied [1], [3], [4], [5], [6], [7], [8], [9], [10] and the literature cited there.

This paper develops a general probabilistic method for constructing positive linear operators useful in the theory of uniform approximation of continuous functions of two variables. We study this sequence of operators by applying the well known theorem of Bohman-Korovkin. Then we evaluate the orders of approximation in terms of the modulus of continuity; in the last section of this paper we present some examples which extend the results of G.C.Jain and S.Pethe [2].

2. Construction of the operators. Let (Ω, A, P) be a probability space in the sence of Kolmogorov, i.e. Ω is an arbitrary abstract space, A is a σ-algebra of subsets of Ω and P

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a probability measure in Ω and on A. Let $(X_n)_{n\geq 1}$, $(Y_m)_{m\geq 1}$ be two real sequences of random variables having the following distributions:

$P(X_n = x_{ni}) = p_{ni}(x,y)$, $(i \in A \subseteq \mathbb{N})$, $P(Y_m = y_{mj}) = q_{mj}(x,y)$, $(j \in B \subseteq \mathbb{N})$ where $(x,y) \in I \times J \subset \mathbb{R} \times \mathbb{R}$; I and J are not necessarily bounded intervals. Let (p_{mj}^{ni}) be defined by:

$$p_{mj}^{ni}(x,y) = P(X_n = x_{ni} \text{ and } Y_m = y_{mj})$$

By the definition of this distribution we have:

$$\sum_{i\in A} p_{mj}^{ni} = q_{mj}, \quad \sum_{j\in B} p_{mj}^{ni} = p_{ni} \quad \text{and consequently} \quad \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} = 1 \tag{1}$$

Now we consider the operators defined by:

$$(L_{mn} f)(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni}(x,y) f(x_{ni}, y_{mj}) \tag{2}$$

where f is a continuous function on $I \times J$.

It is clear that the operators defined in (2) are linear positive. Therefore they are monotone. Let us calculate the values of our operator for the test functions $e_{kl}: D \to \mathbb{R}$, where $e_{kl}(x,y) = x^k y^l$; $k, l \in \{0,1,2\}$, $k+l \leq 2$. Using (1) we have:

$$(L_{mn} e_{00})(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} = 1 = e_{00}(x,y)$$

Let us agree to denote the expectation of the random variable Z by $E(Z)$ and the moments about the origin by $\nu_k(Z)$, the subscript indicating the order of the moment. In these notations:

$$(L_{mn} e_{10})(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} x_{ni} = \sum_{i\in A} p_{ni} x_{ni} = E(X_n)$$

Analogous:

$$(L_{mn} e_{01})(x,y) = E(Y_m)$$

Also:

$$(L_{mn} e_{20})(x,y) = \sum_{i\in A}\sum_{j\in B} p_{mj}^{ni} x_{ni}^2 = \sum_{i\in A}\left(\sum_{j\in B} p_{mj}^{ni}\right) x_{ni}^2 = \sum_{i\in A} p_{ni} x_{ni}^2 = \nu_2(X_n)$$

Analogous: $\left(L_{mn} e_{02}\right)(x,y) = \nu_2\left(Y_m\right)$

At last we compute:

$$\left(L_{mn} e_{11}\right)(x,y) = \sum_{j\in B}\left(\sum_{i\in A} p_{mj}^{ni} x_{ni}\right) y_{mj} = E\left(X_n Y_m\right) \le \left(\nu_2\left(X_n\right)\nu_2\left(Y_m\right)\right)^{1/2},$$

according to the Schwartz inequality. By making use of the results established above, in concordance with the well-known Korovkin's theorem in probabilistic form, we can state:

THEOREM. *Let* $\left(L_{mn}\right)_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ *be introduced in* (2) *and* $f \in C(I\times J)$

If

i) $\lim_n \nu_k\left(X_n\right) = x^k,\ k\in\{1,2\}$

ii) $\lim_m \nu_k\left(Y_m\right) = y^k,\ k\in\{1,2\}$

iii) $\underset{(n,m)}{Lim}\, E\left(X_n Y_m\right) \ge x y$

then we have: $\lim_{(n,m)}\left(L_{mn} f\right) = f$ *uniformly on* $I\times J$.

3. Order of approximation. For evaluating the corresponding orders of approximation, it is convenient to make use of the modulus of continuity of *f*, which is defined by:

$\omega_f\left(\delta_1,\delta_2\right) = \max\left|f(x',y') - f(x'',y'')\right|$, for $|x'-x''| \le \delta_1$, $|y'-y''| \le \delta_2$, δ_1, δ_2 being positive numbers.

We will need the following known property of the modulus of continuity

$$\omega_f\left(\lambda_1\delta_1, \lambda_2\delta_2\right) \le \left(1+\lambda_1+\lambda_2\right)\omega_f\left(\delta_1,\delta_2\right),\quad \left(\lambda_1>0,\ \lambda_2>0\right) \tag{3}$$

We can write succesively:

$$\left|\left(L_{mn}\right)(x,y) - f(x,y)\right| = \left|\sum_{j\in B}\sum_{i\in A} p_{mj}^{ni} f\left(x_{ni}, y_{mj}\right) - f(x,y)\right| \le$$

$$\le \sum_{j\in B}\sum_{i\in A} p_{mj}^{ni}\left|f\left(x_{ni}, y_{mj}\right) - f(x,y)\right| \le \sum_{j\in B}\sum_{i\in A} p_{mj}^{ni}\,\omega\left(\left|x_{ni}-x\right|, \left|y_{mj}-y\right|\right).$$

But by using the inequality (3) we have:

and similar results for $E\left(Y_m\right)$ and $\nu_2\left(Y_m\right)$.

$$E\left(X_n Y_m\right) = \sum_{i\geq 1}\sum_{j\geq 1} ij(nm)^{-1} p_{ni}(nx,\gamma) p_{mj}\left(my,\gamma'\right) \to xy,\ (n,m)\to(\infty,\infty)$$

This completes the proof.

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