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APPROXIMATION PROPERTIES OF A CLASS OF OPERATORS OF STANCU-KANTOROVICH TYPE

by

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Summary. In this paper we consider an extension, in the sense of Kantorovich, of a linear positive operator of Bernstein type $L_{m,r}^{\alpha,\beta}$, introduced by D. D. Stancu in the paper [7]. For this extension we establish some quantitative theorems representing estimations of the orders of approximation, by using the first and the second orders modulus of continuity. Also we give an asymptotic estimation, in the sense of Voronowskaja.

1. Introduction

In the paper [7] D. D. Stancu studied the approximation properties of a class of linear positive operator $L_{m,n}^{\alpha,\beta}$, defined for any function $f: [0,1] \to \mathbb{R}$ by the formula

$$(1) \qquad (L_{m,r}^{\alpha,\beta} f)(x) = \sum_{k=0}^{m} w_{m,k,r}(x) f\left(\frac{k+\alpha}{m+\beta}\right),$$

where

$$w_{m,r,k}(x) = \begin{cases} \binom{m-r}{k} x^k (1-x)^{m-r-k+1} & \text{if } 0 \le k < r \\ \binom{m-r}{k} x^k (1-x)^{m-r-k+1} + \\ + \binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k} & \text{if } r \le k \le m-r \\ \binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k} & \text{if } m-r < k \le m \end{cases}$$

r being a non negative integer so that $m > 2r$, while α and β are real parameters satisfying the condition $0 \le \alpha \le \beta$. For $r = 0$ or $r = 1$ this operator reduces to the classical operator B_m of Bernstein, defined by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}. \tag{2}$$

In the same paper it is shown that the operator $L_{m,r}^{\alpha,\beta}$ can be written under the form:

$$(L_{m,r}^{\alpha,\beta} f)(x) = \sum_{k=0}^{m-r} p_{m-r,k}(x) \left[(1-x) f\left(\frac{k+\alpha}{m+\beta}\right) + x f\left(\frac{k+r+\alpha}{m+\beta}\right) \right].$$

The aim of this paper is to consider an integral extension of the operator $L_{m,r}^{\alpha,\beta}$ in the sense of Kantorovich.

2. The operator $S_{m,r}^{\alpha,\beta}$.

This operator is defined by the following relation

$$(S_{m,r}^{\alpha,\beta} f)(x) = (m+\beta) \sum_{k=0}^{m-r} p_{m-r,k}(x) \left[(1-x) \int_{\frac{k+\alpha}{m+\beta}}^{\frac{k+\alpha+1}{m+\beta}} f(t)dt + \right. \tag{3}$$

$$\left. + x \int_{\frac{k+\alpha+r}{m+\beta}}^{\frac{k+\alpha+r+1}{m+\beta}} f(t)dt \right],$$

where the fundamental polynomials $p_{m-r,k}$ are defined at (2), $m > 2r$, $0 \le \alpha \le \beta-1$.

It is easy to see that this operator reduces (in the special case $r = 0$, $\alpha = 0$, $\beta = 1$) to the Kantorovich operator K_m, defined by

$$(K_m f)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt.$$

3. The convergence of the sequence $(S_{m,r}^{\alpha,\beta})$.

We need first to establish the following

LEMMA 1. The next identities

(i) $(S_{m,r}^{\alpha,\beta} e_o)(x) = 1$

(ii) $(S_{m,r}^{\alpha,\beta} e_1)(x) = \frac{m}{m+\beta} x + \frac{2\alpha+1}{2(m+\beta)}$

(iii) $(S_{m,r}^{\alpha,\beta} e_2)(x) = \frac{(m-r)(m+r-1)}{(m+\beta)^2} x^2 + \frac{2(\alpha+1)m+r^2-r}{(m+\beta)^2} x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta)^2}$

hold, where $e_j(t) = t^j$ $(j = 0,1,2)$, for any $t \in$ [illegible]$,1]$.

Proof. We can write successively

$$(S_{m,r}^{\alpha,\beta} e_o)(x) = (m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left\{\frac{1-x}{m+\beta} + \frac{x}{m+\beta}\right\} = 1 = e_o(x)$$

$$(S_{m,r}^{\alpha,\beta} e_1)(x) = (m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left\{\frac{1-x}{2}\,\frac{2k+2\alpha+1}{(m+\beta)^2} + \frac{x}{2}\,\frac{2k+2\alpha+2r+1}{(m+\beta)^2}\right\} =$$

$$= \frac{1}{m+\beta}\sum_{k=0}^{m-r} p_{m-r,k}(x)\left\{rx+k+\alpha+\frac{1}{2}\right\} = \frac{m}{m+\beta} x + \frac{2\alpha+1}{2(m+\beta)},$$

since

$$(4)\qquad \sum_{k=0}^{m-r} kp_{m-r,k}(x) = \sum_{k=1}^{m-r} (m-r)\binom{m-r-1}{k-1}x^{k-1}(1-x)^{m-r-k} = (m-r)x.$$

We notice that:

$$(S_{m,r}^{\alpha,\beta} e_1)(x) - (L_{m,r}^{\alpha,\beta} e_1)(x) = \frac{1}{2(m+\beta)}.$$

For the monomial e_2 we can write successively:

$$(S_{m,r}^{\alpha,\beta} e_2)(x) = (m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[\frac{1-x}{3}\,\frac{(k+\alpha+1)^3-(k+\alpha)^3}{(m+\beta)^3} + \right.$$

$$\left. + \frac{x}{3}\,\frac{(k+r+\alpha+1)^3-(k+r+\alpha)^3}{(m+\beta)^3}\right] = \frac{1}{3(m+\beta)^2}\sum_{k=0}^{m-r} p_{m-r,k}(x)(3k^2+3(2rx+2\alpha+1)k +$$

$$+\ 3(r^2+2r\alpha+r)x+3\alpha^2+3\alpha+1) = \frac{1}{3(m+\beta)^2}(3(2rx+2\alpha+1)(m-r)x+3(r^2+2r\alpha+r)x+$$

$$+\ 3\alpha^2+3\alpha+1+3(m-r)(m-r-1)x^2+3(m-r)x) =$$

$$= \frac{(m-r)(m+r-1)}{(m+\beta)^2}x^2 + \frac{2(\alpha+1)m+r^2-r}{(m+\beta)^2}x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta)^2},$$

according to the relation (4) and also to the next identity:

$$\sum_{k=0}^{m-r} k^2 \binom{m-r}{k} x^k (1-x)^{m-r-k} = (m-r)x\sum_{k=1}^{m-r} k\binom{m-r-1}{k-1}x^{k-1}(1-x)^{m-r-k} =$$

$$= (m-r)x\left[\sum_{k=2}^{m-r}(k-1)\binom{m-r-1}{k-1}x^{k-1}(1-x)^{m-k-r} + \sum_{k=1}^{m-r}\binom{m-r-1}{k-1}x^{k-1}(1-x)^{m-r-k}\right] =$$

$$= (m-r)x((m-r-1)x+1).$$

This completes the proof of our Lemma 1.

Now we state and prove

THEOREM 1. If $f \in C[0,1]$ then the sequence $(S_{m,r}^{\alpha,\beta} f)$ converges to f uniformly on $[0,1]$.

Proof. By making use of the identities (i), (ii), (iii), we can write

$$\lim_{m\to\infty}(S_{m,r}^{\alpha,\beta}\, e_j)(x) = e_j(x), \qquad (j = 0,1,2)$$

uniformly on $[0,1]$. Consequently our assertion follows directly from the well-known theorem of Bohman Korovkin.

4. Estimation of the order of approximation.

In this section we are concerned with the estimation of the order of approximation of a function $f \in C[0,1]$ by the operator (3). It may be simply described by means of the modulus of continuity, defined by:

$$\omega(\delta) = \omega(f,\delta) = \sup|f(x'')-f(x')|,$$

where x' and x'' are points from $[0,1]$ so that $|x''-x'| < \delta$, δ being a positive number.

Our next lemma gives the representation for $(S_{m,r}^{\alpha,\beta}\varphi_x^2)$, where $\varphi_x(t) = |t-x|$, $0 \le t \le 1$.

LEMMA 2. For the operator $S_{m,r}^{\alpha,\beta}$ the following equality

$$(S_{m,r}^{\alpha,\beta}\varphi_x^2)(x) = \frac{-m+\beta^2+r-r^2}{(m+\beta)^2}x^2 + \frac{m+r^2-r-\beta(2\alpha+1)}{(m+\beta)^2}x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta)^2}$$

holds.

Proof. Because $(S_{m,r}^{\alpha,\beta}\varphi_x^2)(x) = (S_{m,r}^{\alpha,\beta}e_2)(x)-2x(S_{m,r}^{\alpha,\beta}e_1)(x)+x^2$, it is easy to see that Lemma 1 implies the desired result.

Further we state and prove

THEOREM 2. If we assume that

$$m+r^2-r > \beta^2 > (2\alpha+1)^2, \tag{5}$$

then for each $f \in C[0,1]$ we have

$$\|S_{m,r}^{\alpha,\beta} f - f\| \le \left(\frac{3}{2} + O\left(\frac{1}{m}\right)\right)\omega\left(\frac{1}{\sqrt{m+\beta}}\right). \tag{6}$$

Proof. We can write successively

$$|(S_{m,r}^{\alpha,\beta} f)(x)-f(x)|=\left|(m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)\int_{\frac{k+\alpha}{m+\beta}}^{\frac{k+\alpha+1}{m+\beta}}(f(t)-f(x))dt\ +\right.\right.$$

$$\left.\left.+\ x\int_{\frac{k+\alpha+r}{m+\beta}}^{\frac{k+\alpha+r+1}{m+\beta}}(f(t)-f(x))dt\right]\right| \le$$

$$\le(m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)\int_{\frac{k+\alpha}{m+\beta}}^{\frac{k+\alpha+1}{m+\beta}}|f(t)-f(x)|dt+\int_{\frac{k+\alpha+r}{m+\beta}}^{\frac{k+\alpha+r+1}{m+\beta}}|f(t)-f(x)|dt\right].$$

Using the following known properties of the modulus of continuity:

$$|f(x'')-f(x')| \le \omega(|x''-x'|), \qquad \omega(\lambda\delta) \le ([\lambda]+1)\omega(\delta) \le (\lambda+1)\omega(\delta),$$

where $\lambda > 0$, we obtain

$$|f(t)-f(x)| \le \omega(|t-x|) = \omega\left(\frac{1}{\delta}|t-x|\delta\right) \le \left(1 + \frac{1}{\delta}|t-x|\right)\omega(\delta).$$

Consequently we can write

$$|(S_{m,r}^{\alpha,\beta} f)(x)-f(x)| \le \left(1 + \frac{1}{\delta}(S_{m,r}^{\alpha,\beta}\varphi_x)(x)\right)\omega(\delta).$$

According to Lemma 1 and to the fact that

$$(S_{m,r}^{\alpha,\beta}\varphi_x)(x) \le \left[(S_{m,r}^{\alpha,\beta}e_o)(x)(S_{m,r}^{\alpha,\beta}\varphi_x^2)(x)\right]^{1/2}, \tag{7}$$

we obtain

$$|(S_{m,r}^{\alpha,\beta} f)(x)-f(x)| \le \left(1 + \frac{1}{\delta}(S_{m,r}^{\alpha,\beta}\varphi_x^2)^{1/2}(x)\right)\omega(\delta).$$

If in the Lemma 2 we substitute $m+r^2-r = p$, p being a natural number, we can write further:

$$(S_{m,r}^{\alpha,\beta}\varphi_x^2)(x) = \frac{1}{(m+\beta)^2}\left((\beta^2-p)x^2+(p-2\alpha\beta-\beta)x+\alpha^2+\alpha + \frac{1}{3}\right).$$

If we assume that $p^{1/2} > \beta > 2\alpha+1$, then we have:

$$\max_{0\le x\le 1}\left((\beta^2-p)x^2+(p-2\alpha\beta-\beta)x\right) = \frac{(p-2\alpha\beta-\beta)^2}{4(p-\beta^2)} = \frac{p-\beta^2}{4} + \frac{(\beta^2-2\alpha\beta-\beta)^2}{4(p-\beta^2)} - \frac{1}{2}(\beta^2-2\alpha\beta-\beta).$$

By using this result, we can deduce

$$|(S_{m,r}^{\alpha,\beta} f)(x)-f(x)| \le$$

$$\le\left\{1+\frac{1}{2\delta(m+\beta)}\sqrt{m+r^2-r-\beta^2+\frac{\beta^2(\beta-2\alpha-1)^2}{m+r^2-r-\beta^2}+(2\alpha+1)^2-2\beta(\beta-2\alpha-1)+\frac{1}{3}}\right\}\omega(\delta)\le$$

$$\le\left\{1+\frac{1}{2\delta(m+\beta)}\sqrt{m+r^2-r+\frac{\beta^2(\beta-2\alpha-1)^2}{m+r^2-r-\beta^2}+\frac{1}{3}}\right\}\omega(\delta).$$

Inserting $\delta = \frac{1}{\sqrt{m+\beta}}$ and using the notation $c_{m,r}^{(\alpha,\beta)} = -\beta+r^2-r+\frac{\beta^2(\beta-2\alpha-1)^2}{m+r^2-r-\beta^2}+\frac{1}{3}$, we get to the following final result:

$$|(S_{m,r}^{\alpha,\beta} f)(x)-f(x)|\le\left\{1+\frac{1}{2}\sqrt{1+\frac{c_{m,r}^{(\alpha,\beta)}}{m+\beta}}\right\}\omega\left(\frac{1}{\sqrt{m+\beta}}\right)\le\left(\frac{3}{2}+O\left(\frac{1}{m}\right)\right)\omega\left(\frac{1}{\sqrt{m+\beta}}\right).$$

Using the maximum norm over $[0,1]$, we arrive at the inequality (6).

Now we are going to give estimates involving the first order modulus of continuity of the first derivate f'. All our estimates are based upon the following

THEOREM ([1]).

Let L be a linear positive operator mapping $C[a,b]$ into $C[c,d]$, where $[c,d] \subseteq [a,b]$. If $f \in C^1[a,b]$ and if $\omega_1(f',\cdot)$ denotes the first order modulus of continuity of f', then for all $x \in [c,d]$ the estimation

(8) $$|L(f,x)-f(x)| \le |f(x)|\,|L(e_o,x)-1|+\|f'\|\,|L(e_1-x,x)|+\left[L(|e_1-x|,x)+\frac{1}{2h}L((e_1-x)^2,x)\right]\omega_1(f',h)$$

holds, where h is an arbitrarily positive number and $\|\cdot\|$ denotes the sup-norm over $[a,b]$.

From Lemma 1 there follows that the operator $S_{m,r}^{\alpha,\beta}$ satisfies the relations:

$$(S_{m,r}^{\alpha,\beta} e_o)(x) = 1 \text{ and } (S_{m,r}^{\alpha,\beta}(e_1-x))(x) = \frac{2\alpha+1}{2(m+\beta)} - \frac{\beta}{m+\beta}x.$$

As it can be seen from the proof of Theorem 2 we can give the estimation

$$\left(S_{m,r}^{\alpha,\beta}\varphi_x^2\right)^{1/2}(x) \le \frac{1}{2(m+\beta)}\sqrt{m+r^2-r+\frac{\beta^2(\beta-2\alpha-1)^2}{m+r^2-r-\beta^2}+\frac{1}{3}}\,.$$

Since $m > 2r$, we can write:

$$\left(S_{m,r}^{\alpha,\beta}\varphi_x^2\right)^{1/2}(x) \le \frac{1}{2}\lambda_m,$$

where

(9) $\lambda_m = \frac{1}{m+\beta}\sqrt{m+d_r(\alpha,\beta)}$ and $d_r(\alpha,\beta) = r^2-r+\frac{\beta^2(\beta-2\alpha-1)^2}{r^2+r-\beta^2}+\frac{1}{3}$.

Taking (7) into account, the relation (8) leads us to the inequality

$$|(S_{m,r}^{\alpha,\beta}f)(x)-f(x)| \le \|f'\|\left|-\frac{\beta}{m+\beta}x+\frac{2\alpha+1}{2(m+\beta)}\right|+\left\{1+\frac{1}{4h}\lambda_m\right\}\frac{1}{2}\lambda_m\omega_1(f',h).$$

By choosing $h = \frac{1}{4}\lambda_m$, we can state

THEOREM 3. If f has a bounded uniformly continuous derivative on $[0,1]$, then we have:

$$|(S_{m,r}^{\alpha,\beta}f)(x)-f(x)| \le \frac{\|f'\|}{m+\beta}\left|\beta x-\alpha-\frac{1}{2}\right| + \lambda_m\omega_1\left(f',\frac{1}{4}\lambda_m\right),$$

where λ_m is defined at (9).

Now we shall give an estimation of the order of approximation by using the second order modulus of continuity $\omega_2(f,\cdot)$, which is defined by

$$\omega_2(f,h) = \sup\{|f(x-\delta)-2f(x)+f(x+\delta)|:\ x, x\pm\delta \in [0,1],\ 0 < \delta \le h\}.$$

Our estimate will be deduced by means of the following

THEOREM ([2]). Let $[a,b]$ denote a compact interval of the real axis. If $L\colon C[a,b] \to C[c,d]$ (where $[c,d] \subseteq [a,b]$) is a positive linear operator, then for all $f \in C[a,b]$, all $x \in [c,d]$ and each $h > 0$, the following inequality holds:

$$|L(f;x)-f(x)| \le \left\{3\|L\| + \max\left\{\frac{1}{h^2},\frac{1}{(b-a)^2}\right\}L((e_1-x)^2,x)\right\}\omega_2(f,h) + $$
$$+\ 2\max\left\{\frac{1}{h},\frac{1}{b-a}\right\}|L(e_1-x,x)|\omega_1(f,h)+|L(e_o,x)-e_o(x)|\,\|f\|.$$

In our case $\|S_{m,r}^{\alpha,\beta}\| = \sup\{\|S_{m,r}^{\alpha,\beta}f\|:\ \|f\| \le 1\} = 1$ and we have the following explicit evaluation

$$|(S_{m,r}^{\alpha,\beta}f)(x)-f(x)| \le \left\{3+\max\left\{\frac{1}{h^2},1\right\}\right\}\left\{\frac{-m+\beta^2+r-r^2}{(m+\beta)^2}x^2+\frac{m+r^2-r-\beta(2\alpha+1)}{(m+\beta)^2}x+\right.$$

$$+\frac{3\alpha^2+3\alpha+1}{3(m+\beta)^2}\Big]\omega_2(f,h) + 2\max\left\{\frac{1}{h},1\right\}\left|\frac{2\alpha+1}{2(m+\beta)} - \frac{\beta}{m+\beta}x\right|\omega_1(f,h).$$

For $h = \dfrac{1}{\sqrt{m+\beta}}$ this becomes:

$$|(S_{m,r}^{\alpha,\beta}f)(x)-f(x)| \le \frac{3+m+\beta}{(m+\beta)^2}\Big\{(-m+\beta^2+r-r^2)x^2+(m+r^2-r-\beta(2\alpha+1))x+\alpha^2+\alpha+$$

$$+\frac{1}{3}\Big\}\omega_2\left(f,\frac{1}{\sqrt{m+\beta}}\right)+\frac{1}{\sqrt{m+\beta}}|2\beta x-2\alpha-1|\,\omega_1\left(f,\frac{1}{\sqrt{m+\beta}}\right).$$

Under the conditions (5) this result can be expressed by

THEOREM 4. For the operator $S_{m,r}^{\alpha,\beta}$, $f \in C[0,1]$ and $(m+r^2-r)^{1/2} > \beta > 2\alpha+1$, the estimation

$$|(S_{m,r}^{\alpha,\beta}f)(x)-f(x)| \le \frac{3+m+\beta}{4(m+\beta)^2}\Big(m+d_r(\alpha,\beta)\Big)\omega_2\left(f,\frac{1}{\sqrt{m+\beta}}\right)+$$

$$+\frac{1}{\sqrt{m+\beta}}|2\beta x-2\alpha-1|\,\omega_1\left(f,\frac{1}{\sqrt{m+\beta}}\right)$$

holds, where $d_r(\alpha,\beta)$ is defined at (9).

5. An asymptotic estimation.

We shall end this paper by presenting a corresponding Voronowskaja type theorem

THEOREM 5. If the function f is integrable over [0,1] and possesses a second derivative at a point x of [0,1], then we have

$$\lim_{m\to\infty} m((S_{m,r}^{\alpha,\beta}f)(x)-f(x)) = \left(-\beta x+\alpha+\frac{1}{2}\right)f'(x) + \frac{x(1-x)}{2}f''(x).$$

Proof. Let $0 \le t \le 1$. It is well known that if f has a finite second order derivative at a point $x \in [0,1]$ then $f(t)$ can be expanded by Taylor's formula

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + (t-x)^2\varepsilon(t),$$

where ε is a certain real-valued function defined on [0,1] and having the property: $\varepsilon(t) \to 0$ as $t \to x$.

Then by virtue of the linearity of the operator $S_{m,r}^{\alpha,\beta}$ and by Lemma 1 we obtain:

$$(S_{m,r}^{\alpha,\beta}f)(x) = f(x)+f'(x)(S_{m,r}^{\alpha,\beta}(e_1-x))(x) + \frac{f''(x)}{2}(S_{m,r}^{\alpha,\beta}\varphi_x^2)(x)+E_m(x),$$

where $E_m(x)$ is given by

$$E_m(x) = (m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)\int_{\frac{k+\alpha}{m+\beta}}^{\frac{k+1+\alpha}{m+\beta}} (t-x)^2\varepsilon(t)dt + \right.$$

$$\left. + x\int_{\frac{k+\alpha+r}{m+\beta}}^{\frac{k+\alpha+r+1}{m+\beta}} (t-x)^2\varepsilon(t)dt\right].$$

Consequently we can write

$$\lim_{m\to\infty} m((S_{m,r}^{\alpha,\beta} f)(x)-f(x))=\left\{-\beta x+\alpha+ \frac{1}{2}\right\}f'(x) + \frac{f''(x)}{2}(-x^2+x)+\lim_{m\to\infty} mE_m(x).$$

Since $\varepsilon(t)$ tends to zero when t tends to x, it follows that for every $\tau > 0$ there exists an $\delta > 0$ so that for every t, for which $|t-x| < \delta$, we have $|\varepsilon(t)| < \tau$.

Since ε is bounded on $[0,1]$, there exists a constant $M > 0$ so that for every t, for which $|t-x| \geq \delta$, we have

$$|\varepsilon(t)| \leq M \leq M\delta^{-2}(t-x)^2.$$

Then the inequality

$$|\varepsilon(t)| \leq \tau + M\delta^{-2}(t-x)^2$$

holds for every t.

By choosing $\tau = \frac{1}{m}$, we can write:

$$|mE_m(x)|\leq(S_{m,r}^{\alpha,\beta} \varphi_x^2)(x)+m(m+\beta)\sum_{k=0}^{m-r} p_{m-r,k}(x)M\delta^{-2}\left[(1-x)\int_{\frac{k+\alpha}{m+\beta}}^{\frac{k+\alpha+1}{m+\beta}} (t-x)^4dt+\right.$$

$$\left. + x\int_{\frac{k+\alpha+r}{m+\beta}}^{\frac{k+\alpha+r+1}{m+\beta}} (t-x)^4dt\right] = (S_{m,r}^{\alpha,\beta} \varphi_x^2)(x) + mM\delta^{-2}(S_{m,r}^{\alpha,\beta} \varphi_x^4)(x).$$

Using a standard method, the sum from the second member vanishes when m tends to ∞. This completes the proof of this theorem.

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Note. This paper is in final form and no version of it will be submitted for publication elsewhere.