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Foundations of the Continuation Principles of Leray-Schauder Type

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1. The famous continuation theorem of Leray-Schauder for equations in Banach spaces ensures, in its simplest form, the solvability of the equation $x = T(x)$ with T completely continuous , when the possible solutions of the homotopic equations $x = tT(x)$, $t \in [0, 1]$ are a priori bounded independently of t. A more general statement of the Leray-Schauder principle is the following:

Proposition 1 (Leray-Schauder[3]). *Let E be a real Banach space, $U \subset E$ an open bounded set and let $h : [0, 1] \times \overline{U} \to E$ be compact (continuous with $h([0, 1] \times \overline{U})$ relatively compact). Assume the following conditions are satisfied:*

(i) $h(t, x) \neq x$ for all $t \in [0, 1]$ and $x \in \partial U$;

(ii) deg(I-h(0,.), U, 0)$\neq 0$.

Then there exists at least one zero of I-h(1,.) in U. Moreover,

$$(1) \qquad deg(I - h(1, .), U, 0) = deg(I - h(0, .), U, 0).$$

(We denote by deg(f, U, 0) the Leray-Schauder degree of the map f with respect to U and 0).

There is also known the following version of the Leray-Schauder principle, due to Granas [2], in terms of essential mappings instead of topological degree:

Let K be a convex subset of a real Banach space E, $U \subset K$ be bounded and open in K and let $\overline{U}$ and ∂U denote, respectively, the closure and the boundary of U in K. A compact mapping $f : \overline{U} \to K$ is said to be *essential* if it is fixed point free on ∂U and each compact extention g of $f\,|_{\partial U}$ has at least one fixed point in U.

Proposition 2 (Granas [2]). *Let $h : [0, 1] \times \overline{U} \to K$ be compact such that*

(i) $h(t, x) \neq x$ for all $t \in [0, 1]$ and $x \in \partial U$;

(ii) h(0,.) is essential.

Then there exists at least one zero of I-h(1,.) in U. Moreover, h(1,.) is essential too.

In addition, there are known a lot of extentions of Propositions 1 and 2 for single or set-valued mappings with properties more general than the compactness (see [4]). So, a natural question is: what is the common base of all such continuation theorems ? It is the aim of this paper to formulate an answer to this question. As we shall see, at this level of foundations, the concepts of "set", "mapping" and "extention of Urysohn type" will be enough.

2. Let X and Y be two sets and A and B two proper subsets of X and Y respectively. Consider a mapping $H : [0,1] \times X - Y$ and a family M whose elements are functions from X into $[0,1]$ which are constant on A. Also consider a function d which is defined at least on the following family of subsets of X:

$$\{H(a(\cdot),\cdot)^{-1}(B);\ a \in M\} \cup \{\emptyset\}.$$

The nature of the values of d does not import. Denote

$$Z = \{x \in X : H(t,x) \in B \text{ for some } t \in [0,1]\},$$

$F_0 = H(0,\cdot)$, $F_1 = H(1,\cdot)$ and assume the constant functions 0 and 1 belong to M.

Our main result is the following abstract continuation theorem:

Theorem 1 *Assume that the following conditions are satisfied:*

(i) for each $a \in M$ there exists $a^* \in M$ such that

$$a^*(x) = \begin{cases} a(x) & \text{for all } x \in Z \\ 0 & \text{for all } x \in A; \end{cases}$$

(ii) the mapping $F = F_0$ satisfies

$$(2) \qquad d(H(a(\cdot),\cdot)^{-1}(B)) = d(F^{-1}(B)) \neq d(\emptyset),$$

for any $a \in M$ such that

$$H(a(\cdot),\cdot)\,|_A = F\,|_A\,.$$

Then there exists at least one $x \in X - A$ solution to $H(1,x) \in B$. Moreover, $F = F_1$ also satisfies (2) and

$$(3) \qquad d(F_0^{-1}(B)) = d(F_1^{-1}(B)).$$

Proof. We start with the proof of (3). For this, let a^* be associated to the constant 1, according to (i). Obviously, we have

$$H(a^*(\cdot),\cdot)\,|_A = F_0\,|_A$$
$$H(a^*(\cdot),\cdot)^{-1}(B) = F_1^{-1}(B).$$

By (ii), these yield

$$d(F_0^{-1}(B)) = d(H(a^*(\cdot),\cdot)^{-1}(B)) = d(F_1^{-1}(B)),$$

that is (3).

Further, we shall prove that F_1 satisfies (2). To this end, let us consider any $a \in M$ such that $H(a(\cdot),\cdot)\,|_A = F_1\,|_A$. We discuss two cases:

Case 1. Assume $F_0 |_A = F_1 |_A$. Then, $H(a(\cdot), \cdot) |_A = F_0 |_A$ and hence, since F_0 satisfies (2) and we have (3), we deduce

$$d(H(a(\cdot), \cdot)^{-1}(B)) = d(F_0^{-1}(B)) = d(F_1^{-1}(B)),$$

as desired.

Case 2. Assume $F_0 |_A \neq F_1 |_A$. Take any $a^* \in M$ associated to a according to (i). One has

$$H(a^*(\cdot), \cdot) |_A = F_0 |_A,$$

whence, since F_0 satisfies (2), it follows that

$$d(H(a^*(\cdot), \cdot)^{-1}(B)) = d(F_0^{-1}(B)).$$

On the other hand,

$$H(a^*(\cdot), \cdot)^{-1}(B) = H(a(\cdot), \cdot)^{-1}(B).$$

Hence

$$d(H(a^*(\cdot), \cdot)^{-1}(B)) = d(H(a(\cdot), \cdot)^{-1}(B)).$$

These, together with (3), yield

$$d(H(a(\cdot), \cdot)^{-1}(B)) = d(F_0^{-1}(B)) = d(F_1^{-1}(B)).$$

Thus, F_1 satisfies (2) and the proof is complete.

A mapping F of the form $H(a(\cdot), \cdot)$ with $a \in M$, satisfying (2), is said to be *d-essential*. If F is a d-essential mapping, then $F^{-1}(B) \neq \emptyset$, i.e. there exists at least one solution to the inclusion $F(x) \in B$. Theorem 1 saies that, under assumption (i), the d-essentiality of $F_0 = H(0, \cdot)$ spreads to $F_1 = H(1, \cdot)$. It is easy to see that, if $1 - a \in M$ for any $a \in M$, then the d-essentiality of F_0 is in fact equivalent to the d-essentiality of F_1. For the proof, apply Theorem 1 once again, to $\overline{H}(t, x) = H(1 - t, x)$. If in addition, $ta \in M$ for all $a \in M$ and $t \in [0, 1]$, then the mappings $H(t, \cdot)$, $t \in [0, 1]$, are all d-essential or all d-inessential.

In applications, Theorem 1 is used to prove the existence of solutions to the inclusion $F_1(x) \in B$, when it is known that F_0 is d-essential. It follows that it is important to have methods to identify the d-essentiality property. Such methods arise from the *fixed point theory* and *degree theory*. The function d is, for first kind methods, the simple indicator function taking only two values:

$$d(C) = \begin{cases} 1 & \text{if } \emptyset \neq C \subset X \\ 0 & \text{if } C = \emptyset \end{cases}$$

while for the second kind methods, its values are integers which are obtained by means of the degree (see [6], [7]).

As regard condition (i), notice that, since $1 \in M$, there is $a^* \in M$ such that $a^* = 1$ on Z and $a^* = 0$ on A. Thus, (i) implies $Z \cap A = \emptyset$. Condition (i) is, at least in some particular cases, in connexion with the extension theorems of Urysohn type from the general topology. Here are two such particular cases for normal and completely regular spaces (for another example we send to [5]).

1) The continuation principle in normal spaces.

A special version of Theorem 1 is obtained if we assume that X is a normal topological space, H is continuous and

$$M = \{a \in C(X;[0,1]) : a\,|_A \text{ is constant }\}.$$

Then the condition (i) is satisfied provided that

$$(i') \qquad \overline{Z} \cap \overline{A} = \emptyset.$$

Indeed, if we assume (i'), then, by the Urysohn's characterization of normality (see [1]), there is $\theta \in C(X;[0,1])$ such that $\theta(x) = 0$ on $\overline{A}$ and $\theta(x) = 1$ on $\overline{Z}$. Now, if we take any $a \in M$, we see that $a^*(x) = \theta(x)a(x)$ has the required properties in (i). Thus (i') implies (i).

Recall that any metric space is normal.

2) The continuation principle in completely regular spaces.

Assume, in particular, X is completely regular, i.e. for every $p \in X$ and every closed subset $D \subset X$, $p \notin D$, there exists $\theta \in C(X;[0,1])$ such that $\theta(p) = 1$ and $\theta(x) = 0$ on D. Take H and M as in the previous example. Then, a sufficient condition for (i) is

$$(i'') \qquad \overline{Z} \bigcap \overline{A} = \emptyset \text{ and } \overline{Z} \text{ or } \overline{A} \text{ is compact.}$$

This follows from a Urysohn type characterization of the completely regular spaces (see [1, Problem XI.2.11]).

Notice that any Hausdorff locally convex space is completely regular.

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