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AN APPLICATION OF DIVIDED DIFFERENCES

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Abstract. By using the divided differences as fundamental mathematical tools we investigate the monotonicity property of a sequence of linear positive operators which was introduced in [2] by Bleimann, Butzer and Hahn.

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1. Introduction.

In 1980 Bleimann, Butzer and Hahn [2] constructed a sequence of linear positive operators L_n defined on the space $C[0,\infty)$ of continuous functions f on the interval $[0,\infty)$. These operators are given by means of the following formula:

$$(L_n f)(x) = (1+x)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k, \qquad n \in \mathbb{N}. \tag{1}$$

It was shown that $(L_n f)(x)$ tends pointwise on $[0,\infty)$ to $f(x)$ for $n \to \infty$, and the convergence is uniform on each compact subset of $[0,\infty)$. Also, estimations for rate of convergence of $|(L_n f)(x)-f(x)|$ were established, measured in terms of the second modulus of continuity of f. It is easy to see that the operators L_n are intimately related to the well-known Bernstein operators. Indeed, by the rational transformation $y = \frac{x}{1+x}$, the expression $(L_n f)$ becomes:

$$\sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} F\left(\frac{k}{n+1}\right), \qquad n \in \mathbb{N}, \tag{2}$$

with $F(y) = f\left(\frac{y}{1-y}\right)$. So, (2) represents the ordinary Bernstein operator B_n with slightly modified nodes.

These last years, several authors [1], [4], [5], have studied this operator and descovered interesting properties. Also, it was extended at two variables, see [3].

The purpose of this note is to determine the difference of two consecutive terms of the sequence $L_n f$, by using the divided differences. This way we are able to find information about the monotonicity of this sequence.

2. Results.

It is obvious that (1) can be written as follows:

$$(L_n f)(x) = \frac{1}{(x+1)^{n+1}} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} x^k + \frac{1}{(x+1)^{n+1}} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} x^{k+1}.$$

In the last sum of the equality we set $i = k+1$ and then, denoting again the summation index by k, we obtain:

$$(L_n f)(x) = \frac{1}{(x+1)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n-k+1}\right) x^k + \frac{1}{(x+1)^{n+1}} \sum_{k=1}^{n+1} \binom{n}{k-1} f\left(\frac{k-1}{n-k+2}\right) x^k =$$

$$= \frac{f(0)}{(x+1)^{n+1}} + \frac{f(n)}{(x+1)^{n+1}} x^{n+1} + \frac{1}{(x+1)^{n+1}} \sum_{k=1}^{n} \left(\binom{n}{k} f\left(\frac{k}{n-k+1}\right) + \binom{n}{k-1} f\left(\frac{k-1}{n-k+2}\right)\right) x^k. \qquad (3)$$

Clearly:

$$(L_{n+1} f)(x) = \frac{f(0)}{(x+1)^{n+1}} + \frac{1}{(x+1)^{n+1}} \sum_{k=1}^{n} \binom{n+1}{k} f\left(\frac{k}{n-k+2}\right) x^k + \frac{f(n+1)}{(x+1)^{n+1}} x^{n+1} \qquad (4)$$

By using (3) and (4), we obtain:

$$(L_{n+1} f)(x) - (L_n f)(x) = \left(\frac{x}{x+1}\right)^{n+1} (f(n+1) - f(n))$$

$$+ \frac{1}{(x+1)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} \left\{\frac{n+1}{n-k+1} f\left(\frac{k}{n-k+2}\right) - f\left(\frac{k}{n-k+1}\right) - \frac{k}{n-k+1} f\left(\frac{k-1}{n-k+2}\right)\right\} x^k. \qquad (5)$$

Starting from the recurrence formula of divided differences:

$$[x,y,z;f] = \frac{[y,z;f]-[x,y;f]}{z-x}$$

and choosing $x = \frac{k-1}{n-k+2}$, $y = \frac{k}{n-k+2}$, $z = \frac{k}{n-k+1}$ it follows that:

$$\frac{n+1}{n-k+1} f\left(\frac{k}{n-k+2}\right) - f\left(\frac{k}{n-k+1}\right) - \frac{k}{n-k+1} f\left(\frac{k-1}{n-k+2}\right) =$$

$$= -\frac{k(n+1)}{(n-k+1)^2(n-k+2)^2}\left[\frac{k-1}{n-k+2}, \frac{k}{n-k+2}, \frac{k}{n-k+1}; f\right]. \tag{6}$$

On the other hand:

$$\frac{k}{(n-k+1)(n-k+2)}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k-1}, \quad 1 \le k \le n. \tag{7}$$

Substituting (6) and (7) in the relation (5), we get:

$$(L_{n+1}f)(x)-(L_nf)(x) = \left(\frac{x}{x+1}\right)^{n+1}(f(n+1)-f(n)) -$$

$$-\frac{1}{(x+1)^{n+1}}\sum_{k=1}^{n}\frac{1}{(n-k+1)(n-k+2)}\binom{n+1}{k-1}\left[\frac{k-1}{n-k+2}, \frac{k}{n-k+2}, \frac{k}{n-k+1}; f\right]x^k =$$

$$= \left(\frac{x}{x+1}\right)^{n+1}(f(n+1)-f(n)) -$$

$$-\frac{1}{(x+1)^{n+1}}\sum_{k=0}^{n-1}\frac{1}{(n-k)(n-k+1)}\binom{n+1}{k}\left[\frac{k}{n-k+1}, \frac{k+1}{n-k+1}, \frac{k+1}{n-k}; f\right]x^{k+1}.$$

Now, we can state the following proposition:

Theorem A. Let $L_n\colon C[0,\infty) \longrightarrow C[0,\infty)$ be defined at (1). Then, the identity

$$(L_{n+1}f)(x)-(L_nf)(x) =$$

$$= \frac{1}{(x+1)^{n+1}}\left\{[n,n+1;f]x^{n+1} - \sum_{k=0}^{n-1}\alpha_{nk}\left[\frac{k}{n-k+1}, \frac{k+1}{n-k+1}, \frac{k+1}{n-k}; f\right]x^{k+1}\right\}$$

holds, where the coefficients α_{nk} are given below

$$\alpha_{nk} = \frac{1}{(n-k)(n-k+1)}\binom{n+1}{k}, \quad 0 \le k \le n-1. \tag{8}$$

Let $C_B[0,\infty)$ be the space of continuous bounded functions on $[0,\infty)$. Now, we assume that $f \in C_B[0,\infty)$. This supposition implies:

$$\lim_{\lambda\to\infty} \lambda[x,y,\lambda;f] = -[x,y;f]. \tag{9}$$

Consequently we can write:

$$\lambda[x,y,\lambda;f] = \frac{\lambda}{\lambda-x}\left[\frac{f(y)-f(\lambda)}{y-\lambda} - [x,y;f]\right].$$

This relation and the fact that it exists c with the property $|f(t)| \le c$ for any $t \ge 0$ lead us to the desired result.

Replacing (9) in Theorem A we obtain:

Theorem B. Let $L_n: C[0,\infty) \to C[0,\infty)$ be defined at (1). If $f \in C_B[0,\infty)$ the identity

$$(L_{n+1}f)(x)-(L_nf)(x) = -\frac{1}{(x+1)^{n+1}}\left\{\lim_{\lambda\to\infty} \lambda[n,n+1,\lambda;f]x^{n+1} + \right.$$

$$\left. + \sum_{k=0}^{n-1} \alpha_{nk}\left[\frac{k}{n-k+1},\frac{k+1}{n-k+1},\frac{k+1}{n-k};f\right]x^{k+1}\right\}$$

holds, where the coefficients α_{nk} are given at (8).

3. Conclusions.

Firstly, we recall the definition of the notion of higher order convex (concav) function which was introduced by Tiberiu Popoviciu [6].

Definition. A real-valued function defined on an interval I is called convex (concav) of order n on I if all its divided differences of order n+1, on n+2 distinct points of I, are positive (negative).

Naturally, if f is a convex (concav) function of order 0 this means that f is increasing (decreasing).

According to Theorem A and taking into account that $\alpha_{nk} > 0$ for $0 \le k \le n-1$, we can state:

Corollary A. Let the sequence $(L_n)_{n\ge0}$ be defined at (1).

(i) If the function $f \in C[0,\infty)$ is convex of first order and decreasing then the sequence $(L_nf)_{n\ge0}$ is decreasing.

(ii) If the function $f \in C[0,\infty)$ is concav of first order and increasing then the sequence $(L_n f)_{n\geq 0}$ is increasing.

From Theorem B, analysing similarly, we deduce:

Corollary B. Let the sequence $(L_n)_{n\geq 0}$ be defined at (1).

(i) If the function $f \in C_B[0,\infty)$ is convex of first order then the sequence $(L_n f)_{n\geq 0}$ is decreasing.

(ii) If the function $f \in C_B[0,\infty)$ is concav of first order then the sequence $(L_n f)_{n\geq 0}$ is increasing.

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