Existence Results for Nonlinear Boundary Value Problems under Nonresonance Conditions

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ABSTRACT

We give applications of Banach, Schauder, Darbo and Leray-Schauder fixed point theorems to prove existence results for weak solutions of the semilinear Dirichlet problem $-\Delta u - cu = f(x,u,\nabla u)$ in Ω, $u = 0$ on $\partial\Omega$, under the assumption that c is not an eigenvalue of $-\Delta$ and $f(x,u,v)$ has linear growth on u and v. We obtain improvements of some known existence results.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$. We discuss the existence of weak solutions (in $H_0^1(\Omega)$) of the Dirichlet problem

$$-\Delta u - cu = f(x,u,\nabla u)\ ,\ x \in \Omega$$
$$u = 0\ ,\ x \in \partial\Omega \tag{1}$$

where $f : \Omega \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ is a L^2 - Caratheodory function (f is a Caratheodory function, i.e. $f(x,\cdot)$ is continuous for almost all $x \in \Omega$, $f(\cdot,z)$ is measurable for all $z \in \mathbb{R}^{N+1}$, and $f(\cdot,0) \in L^2(\Omega)$) and $c \in \mathbb{R}$.

Denote by λ_n, $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_k \le \dots$, the eigenvalues of $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$, i.e. the numbers for which the problem

$$-\Delta u - \lambda u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

has nontrivial weak solutions.

We shall assume $c \neq \lambda_n$, $n = 1,2,\ldots$ and that

$$|f(x,u,v)| \le a|u| + b|v| + h(x) \tag{2}$$

for all $u \in \mathbb{R}$, $v \in \mathbb{R}^N$ and almost all $x \in \Omega$, where a and b are nonnegative constants and $h \in L^2(\Omega)$.

Problems of this type have been examined by many authors over the last twenty years. We refer to [3], [8] and [10] for results related to ours.

In [10, Section 5] it was studied the existence of $C^\infty(\Omega)$ solutions for a more general problem

$$Lu - A(x,D^\alpha u)u = F(x,D^\alpha u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

with L an elliptic operator of order m and $|\alpha| \le m-1$, where roughly speaking $\mu_j < A(x,D^\alpha u) < \mu_{j+1}$ (μ_j being the eigenvalues of L) and $F(x,z)$ grows *slower* than linearly as $|z| \longrightarrow \infty$.

In [3] the following result was proved via the Schauder fixed point theorem: Assume f, g and $\partial g/\partial u$ satisfy the Caratheodory conditions, there are α, $\alpha_1 \in \mathbb{R}$ and j such that

$$\lambda_j < \alpha \le (\partial g/\partial u)(x,u) \le \alpha_1 < \lambda_{j+1} \tag{3}$$

for $(x,u) \in \Omega \times \mathbb{R}$, and there are $\beta > 0$ and $h(x) \in L^2(\Omega)$ such that

$$|f(x,u,v)|^2 \le \beta^2|v|^2 + h(x)$$

for $(x,u,v) \in \Omega \times \mathbb{R}^{N+1}$. Then, if

$$\beta < (\min\{1-\alpha_1/\lambda_{j+1},\ \alpha/\lambda_j - 1\})/\sqrt{\lambda_1}, \tag{4}$$

there exists at least one solution $u \in H_0^1(\Omega)$ to

$$-\Delta u - g(x,u) = f(x,u,\nabla u),\ x \in \Omega$$
$$u = 0,\ x \in \partial\Omega. \tag{5}$$

As we shall see, in many cases, our results will improve inequality (4).

Recently in [8], it was proved that if $c = 0$ and

$$a/\lambda_1 + b/\sqrt{\lambda_1} < 1 \tag{6}$$

then (1) has via the Schauder fixed point theorem, at least one weak solution. Also, if instead of (2) f satisfies the

Lipschitz condition

$$|f(x,u,v) - f(x,\bar{u},\bar{v})| \leq a|u-\bar{u}| + b|v-\bar{v}| \qquad (7)$$

for all $u,\bar{u} \in \mathbb{R}$, $v,\bar{v} \in \mathbb{R}^N$ and almost all $x \in \Omega$, and if $c = 0$ and (6) holds, then the weak solution to (1) exists and is unique as follows by the contraction mapping principle.

In the present paper we shall obtain an analogue of inequality (6) for the general case of an arbitrary constant c, $c \neq \lambda_n$, $n = 1,2,\ldots$.

For other results and approaches to semilinear boundary value problems we refer to [2], [6], [9] and [11].

2. Preliminaries

Throughout this paper the notations $<\cdot,\cdot>$ and $|\cdot|$ stand for the usual inner product and Euclidian norm in $\mathbb{R}^N$, $(\cdot,\cdot)$ and $\|\cdot\|$ are the inner product and norm in $L^2(\Omega)$ while $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ denote the inner product

$$(u,v)_1 = \int_\Omega <\nabla u, \nabla v> dx \quad (u,v \in H_0^1(\Omega))$$

and the corresponding norm in $H_0^1(\Omega)$ (Recall we have supposed Ω bounded).

We shall use basic facts in the L^2 theory for the linear operator $-\Delta$ subject to the homogeneous Dirichlet boundary contraint:

a) *The Rellich - Kondrachov theorem.* (i) The imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact.

(ii) If in addition Ω has a C^1 boundary, then the imbedding of $H^2(\Omega)$ into $L^2(\Omega)$ is compact (see [4] or [7]).

b) *The inverse of* $-\Delta$. For each $v \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$, $u = (-\Delta)^{-1}v$, to the problem

$$-\Delta u = v \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega,$$

i.e.

$$(u,w)_1 = (v,w) \text{ for all } w \in H_0^1(\Omega). \qquad (8)$$

The linear operator $(-\Delta)^{-1}: L^2(\Omega) \to H_0^1(\Omega)$ is bounded and, by the Rellich-Kondrachov theorem, $(-\Delta)^{-1}$ is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Also, one has

$$1/\lambda_1 = \sup\{\,|(-\Delta)^{-1}v|\ ;\ v \in L^2(\Omega),\ |v| \le 1\}$$

and consequently

$$|(-\Delta)^{-1}v| \le (1/\lambda_1)|v| \quad \text{for all } v \in L^2(\Omega). \tag{9}$$

From (8) and (9) we have

$$\|(-\Delta)^{-1}v\|_1^2 = (v, (-\Delta)^{-1}v) \le (1/\lambda_1)|v|^2,$$

hence

$$\|(-\Delta)^{-1}v\|_1 \le (1/\sqrt{\lambda_1})\|v\| \quad \text{for all } v \in L^2(\Omega). \tag{10}$$

c) *Regularity of weak solutions.* If Ω has a C^2 boundary, then $(-\Delta)^{-1}(L^2(\Omega)) \subset H^2(\Omega)$ and the linear operator $(-\Delta)^{-1}$ is also bounded from $L^2(\Omega)$ into $H^2(\Omega)$ (see [1]).

d) *Eigenfunctions.* There exists a Hilbert base (u_k) in $L^2(\Omega)$ such that $u_k \in H_0^1(\Omega)$ and

$$(u_k, w)_1 = \lambda_k (u_k, w) \quad \text{for all } w \in H_0^1(\Omega),\ k = 1,2,\ldots .$$

Moreover, $(u_k/\sqrt{\lambda_k})$ is a Hilbert base in $(H_0^1(\Omega), (\cdot,\cdot)_1)$ (see [1]).

We shall also use the following well known result:

e) *The Nemitskii superposition operator.* If $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Caratheodory function satisfying the growth condition

$$|g(x,u)| \le C|u| + h(x)$$

for all $u \in \mathbb{R}^m$ and almost all $x \in \Omega$, where $C \ge 0$ and $h \in L^2(\Omega)$, then the mapping

$$u \longrightarrow g(\cdot, u(\cdot))$$

is continuous from $L^2(\Omega;\mathbb{R}^m)$ into $L^2(\Omega)$ (see [9] or [11]).

3. Results

<u>Lemma</u>. Let c be any constant such that $c \ne \lambda_n, n=1,2,\ldots$ Then, for each $v \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ to the problem $-\Delta u - cu = v$ in Ω, $u = 0$ on $\partial\Omega$, which we denote by $L^{-1}v$ (where $Lu = -\Delta u - cu$), and we have the following eigenfunction expansion

$$L^{-1}v = \sum_{k=1}^{\infty}(\lambda_k - c)^{-1}(v,u_k)u_k, \tag{11}$$

where the series converges in $H_0^1(\Omega)$. *In addition,*

$$\|L^{-1}v\| \le \gamma_c\|v\| \quad \text{for all } v\in L^2(\Omega), \tag{12}$$

where $\gamma_c = \max\{1/|\lambda_k - c|;\ k = 1,2,\dots\}$.

<u>*Proof.*</u> We first prove the convergence in $H_0^1(\Omega)$ of series (11). Since $(u_k/\sqrt{\lambda_k})$ is a Hilbert base in $(H_0^1(\Omega),(\cdot,\cdot)_1)$, we have

$$\|\sum_{k=n+1}^{n+p}(\lambda_k-c)^{-1}(v,u_k)u_k\|_1^2 = \sum_{k=n+1}^{n+p}(v,u_k)^2\lambda_k/(\lambda_k-c)^2 \le$$
$$\le C\sum_{k=n+1}^{n+p}(v,u_k)^2$$

where C is such that $\lambda_k/(\lambda_k-c)^2 \le C$ for $k = 1,2,\dots$. Now the convergence of (11) follows from the convergence of the numerical series $\sum_{k=1}^{\infty}(v,u_k)^2$. Let $u \in H_0^1(\Omega)$ be the sum of series (11). Next we check that $Lu = v$ weakly, i.e. $(u,w)_1 - c(u,w) = (v,w)$ for all $w \in H_0^1(\Omega)$. Indeed, we have

$$(u,w)_1 = \sum_{k=1}^{\infty}(\lambda_k-c)^{-1}(v,u_k)(u_k,w)_1 =$$
$$= \sum_{k=1}^{\infty}\lambda_k(\lambda_k-c)^{-1}(v,u_k)(u_k,w)$$

and

$$(u,w) = \sum_{k=1}^{\infty}(\lambda_k-c)^{-1}(v,u_k)(u_k,w).$$

Hence

$$(u,w)_1-c(u,w) = \sum_{k=1}^{\infty}(v,u_k)(u_k,w) =$$
$$= (\sum_{k=1}^{\infty}(v,u_k)u_k,w) = (v,w)$$

as desired.

The uniqueness folows from $c \ne \lambda_n$, $n = 1,2,\dots$.

To prove (12), observe that

$$\|\sum_{k=1}^{n}(\lambda_k-c)^{-1}(v,u_k)u_k\|^2 \longrightarrow \|L^{-1}v\|^2 \quad \text{as } n\to\infty$$

and, on the other hand,

$$\|\sum_{k=1}^{n}(\lambda_k-c)^{-1}(v,u_k)u_k\|^2 = \sum_{k=1}^{n}(\lambda_k-c)^{-2}(v,u_k)^2 \le$$
$$\le \gamma_c^2\sum_{k=1}^{n}(v,u_k)^2 \longrightarrow \gamma_c^2\|v\|^2 \quad \text{as } n\to\infty.$$

The proof of Lemma is thus complete.

<u>*Theorem*</u> 1. *Suppose*

$$\lambda_j < c < \lambda_{j+1} \tag{13}$$

for some $j \in \mathbb{N}$ $(\lambda_0 = -\infty)$ *and that* f *is a* L^2 - *Caratheodory function satisfying* (7) *with two nonnegative constants a and* *b such that*

$$a\gamma_c + b\sqrt{\gamma_c(1+c\gamma_c)} < 1. \tag{14}$$

Then problem (1) *has a unique solution* $u \in H_0^1(\Omega)$.

Proof. Solving (1) is equivalent to finding a fixed point of the following mapping

$$A : L^2(\Omega) \longrightarrow L^2(\Omega),\ A(v) = f(\cdot, L^{-1}v, \nabla L^{-1}v). \tag{15}$$

From (7) and e) it follows that A is well defined.

Now we show that A is in fact a contraction mapping. For this, let $v_1, v_2 \in L^2(\Omega)$. Then

$$\| A(v_1) - A(v_2) \| \le a\| L^{-1}(v_1 - v_2) \| + b\| L^{-1}(v_1 - v_2) \|_1 .$$

From (12) we have

$$\| L^{-1}(v_1 - v_2) \| \le \gamma_c \| v_1 - v_2 \| .$$

On the other hand, using (8) and (12) we get

$$\| L^{-1}(v_1 - v_2) \|_1^2 = c\| L^{-1}(v_1 - v_2) \|^2 + (v_1 - v_2, L^{-1}(v_1 - v_2)) \le$$
$$\le c\gamma_c^2 \| v_1 - v_2 \|^2 + \gamma_c \| v_1 - v_2 \|^2$$

and therefore

$$\| L^{-1}(v_1 - v_2) \|_1 \le \sqrt{\gamma_c(1+c\gamma_c)}\ \| v_1 - v_2 \| . \tag{16}$$

Consequently

$$\| A(v_1) - A(v_2) \| \le (a\gamma_c + b\sqrt{\gamma_c(1+c\gamma_c)})\| v_1 - v_2 \|$$

which, by (14), shows that A is a contraction mapping on $L^2(\Omega)$.

Remark 1. In case $c = 0$ we have $\gamma_c = 1/\lambda_1$ and inequality (14) reduces to (6). Thus, Theorem 1 generalizes Theorem 1 in [8].

Notice that in Theorem 1 no smothness assumption on the boundary $\partial\Omega$ is required.

Theorem 2. Suppose Ω *has a* C^2 *boundary,* (13) *holds and* f *is a Caratheodory function satisfying* (2) *with* a *and* b *as in* (14). *Then problem* (1) *has at least one solution* $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. To solve (1) we look for a fixed point of mapping (15). This time, A is completely continuous. Indeed, due to the C^2 smothness of boundary $\partial\Omega$, $L^{-1}v \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $v \in L^2(\Omega)$. It follows that $\nabla L^{-1}v \in H^1(\Omega;\mathbb{R}^N)$ and so $(L^{-1}v, \nabla L^{-1}v) \in H^1(\Omega;\mathbb{R}^{N+1})$. Next, by Rellich - Kondrachov theorem, the imbedding of $H^1(\Omega;\mathbb{R}^{N+1})$ into $L^2(\Omega;\mathbb{R}^{N+1})$ is compact and since the Nemitskii mapping $f(\cdot, u(\cdot))$ is continuous and bounded from $L^2(\Omega;\mathbb{R}^{N+1})$ into $L^2(\Omega)$, it follows that A is completely continuous from $L^2(\Omega)$ into itself. Similar estimations to those in the proof of Theorem 1 show that

$$\|A(v)\| \le a\|L^{-1}v\| + b\|L^{-1}v\|_1 + \|h\| \le$$
$$\le (a\gamma_c + b\sqrt{\gamma_c(1+c\gamma_c)})\|v\| + \|h\| .$$

By (14), this shows that there exists a sufficiently large closed ball in $L^2(\Omega)$ which is mapped into itself by A and so we may apply Schauder fixed point theorem.

Remark 2. In case $c = 0$ Theorem 2 reduces to Theorem 2 in [8].

Corollary. *Suppose Ω has a C^2 boundary, $f : \Omega \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are L^2 - Caratheodory functions and there are α, α_1, $b \in \mathbb{R}$ and $j \in \mathbb{N}$ such that (3), (13) hold and*

$$|f(x,u,v)| \le b|v| + h(x) \tag{17}$$

for all $u \in \mathbb{R}$, $v \in \mathbb{R}^N$ and almost all $x \in \Omega$, where $h \in L^2(\Omega)$. If inequality (14) is true for $a = \max\{|c-\alpha|, |c-\alpha_1|\}$, then problem (5) has at least one solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Problem (5) is equivalent to

$$-\Delta u - cu = f(x,u,\nabla u) + g(x,u) - cu , \ x \in \Omega$$
$$u = 0, \ x \in \partial\Omega \tag{18}$$

where, by (3), one has

$$|g(x,u) - cu| \le a|u| + |g(x,0)|$$

for all $u \in \mathbb{R}$ and almost all $x \in \Omega$. Now we may apply Theorem 2 to problem (18).

Remark 3. If we denote

$$\delta = (\lambda_{j+1} - \lambda_j)/(\alpha/\lambda_j - \alpha_1/\lambda_{j+1})$$

and we choose

$$c = \lambda_{j+1} - \delta(\alpha/\lambda_j - 1) .$$

then $\lambda_{j+1} - c = \delta(\alpha/\lambda_j - 1)$ and $c - \lambda_j = \delta(1 - \alpha/\lambda_{j+1})$. Thus, the right hand side in (4) equals $(\sqrt{\lambda_1}\delta\gamma_c)^{-1}$. Consequently, (4) is equivalent, via (17), to

$$b\sqrt{\lambda_1}\delta\gamma_c < 1 , \qquad (18)$$

which is in most cases more restrictive than our condition (14). For instance, in case that $\alpha = \alpha_1 = (\lambda_j\lambda_{j+1})^{1/2}$ we have $\delta = c = \alpha$, $a = 0$, $\gamma_c = (\alpha - \lambda_j)^{-1}$, and it is easily seen that

$$\sqrt{\lambda_1}\delta\gamma_c > \sqrt{\gamma_c(1+c\gamma_c)}$$

for j large enough.

The smothness of the boundary $\partial\Omega$ was required for the complete continuity of A in case that f depends on ∇u. In the following theorem the mapping A will be only a set-contraction and the smothness assumption on $\partial\Omega$ will be not necessary.

Theorem 3. Suppose (13) *holds and f has the decomposition $f(x,u,v) = f_1(x,u) + f_2(x,u,v)$ with f_1 and f_2 L^2-Caratheodory functions such that*

$$|f_1(x,u)| \le a_1|u| + h(x) \quad (h \in L^2(\Omega))$$

and

$$|f_2(x,u,v) - f_2(x,\bar{u},\bar{v})| \le a_2|u-\bar{u}| + b|v-\bar{v}|$$

for all u, $\bar{u} \in \mathbb{R}$, $v, \bar{v} \in \mathbb{R}^N$ and almost all $x \in \Omega$. If $a = a_1 + a_2$ and b satisfy (14), *then* (1) *has at least one solution $u \in H_0^1(\Omega)$.*

Proof. We have the following decomposition of the mapping A:

$$A = A_1 + A_2 , \; A_1(v) = f_1(\cdot, L^{-1}v) , \; A_2(v) = f_2(\cdot, L^{-1}v, \nabla L^{-1}v) ,$$

where A_1 is completely continuous and A_2 is a contraction mapping. Hence A is a set-contraction, which, by (14), maps a sufficiently large ball of $L^2(\Omega)$ into itself. Thus we may apply Darbo fixed point theorem (see [5, Theorem 2.9.1]) and find a fixed point of A.

Remark 4. If $f = f(u)$ where $f \in C(\mathbb{R})$ and $f(u)/u \longrightarrow 0$ as

$|u| \to \infty$, then we may choose $a_2 = b = 0$ and $a = a_1$ small enough that $a\gamma_c < 1$. Thus, (14) automatically is satisfied and so Proposition 2.7.1 in [9] is a special case of Theorem 3.

We conclude with two results concerning the case where a sign condition is satisfied by a component of f and $c = 0$ (equivalently $c \leq 0$).

Theorem 4. Suppose Ω has a C^2 boundary and f has the decomposition $f(x,u,v) = f_1(x,u,v) + h(x)$ where $h \in L^2(\Omega)$ and f_1 is a Caratheodory function satisfying

$$|f_1(x,u,v)| \leq a|u| + b|v| \tag{20}$$

and

$$u f_1(x,u,v) \leq 0 \tag{21}$$

for all $u \in \mathbb{R}$, $v \in \mathbb{R}^N$ and almost all $x \in \Omega$, with some arbitrary constants a and b. Then problem (1) has for $c = 0$ at least one solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. We look for a fixed point $v \in L^2(\Omega)$ of mapping (15) which, as in the proof of Theorem 2, is completely continuous. Now we show that the set of the solutions to

$$v = \lambda A(v),\ 0 \leq \lambda \leq 1 \tag{22}$$

is bounded. Indeed, let v be any solution to (22) and $u = (-\Delta)^{-1}v$. Then, on using (21), we get

$$\| u\|_1^2 = \lambda(f_1(\cdot,u,\nabla u),u) + \lambda(h,u) \leq \lambda\|h\|\,\|u\|.$$

It follows that there exists a constant R_1 independent of λ with $\|(-\Delta)^{-1}v\|_1 \leq R_1$ for each solution v to (22). This, by (22) and (20), implies that there is R independent of λ such that $\|v\| < R$ for each solution to (22). The conclusion now follows by applying the Leray-Schauder principle for completely continuous mappings (see [5, 5.18.1]).

Theorem 5. Suppose f has the decomposition $f(x,u,v) = f_0(x,u) + f_1(x,u,v)$ where f_0 and f_1 are L^2-Caratheodory functions satisfying

$$|f_0(x,u)| \leq a_0|u|,\ \ u f_0(x,u) \leq 0,$$

$$|f_1(x,u,v)-f_1(x,\bar{u},\bar{v})| \le a_1|u-\bar{u}| + b_1|v-\bar{v}|$$

for all $u,\bar{u}\in\mathbb{R}$, $v,\bar{v}\in\mathbb{R}^N$ and almost all $x\in\Omega$. Then, for c = 0 and sufficiently small a_1 and b_1, (1) has at least one solution $u\in H_0^1(\Omega)$.

<u>*Proof*</u>. First show as in the proof of Theorem 3 that for a_1 and b_1 small enough, A is a set-contraction. Next show the a priori boundedness of all solutions of (22) and finally apply the Leray-Schauder principle for set-contractions.

Notice that in Theorem 5 we need no restrictions on a_1 and the boundary of Ω.

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