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A CLASS OF BLEIMANN, BUTZER AND HAHN TYPE OPERATORS

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Abstract. In this paper we introduce a Bleimann, Butzer and Hahn type operator L_n^a where a is a real and positive parameter. If the classical operators have the nodes $x_k = \dfrac{k}{n-k+1}$ now we take $x_k^a = \dfrac{k+a}{n-k+1}$, $k = 0, 1, \ldots, n$. It is shown that $(L_n^a f)(x)$ tends pointwise on $[0, \infty)$ to $f(x)$ for $n \to \infty$. Moreover, estimations for the rate of convergence of $(L_n^a f)(x) - f(x)$ are established.

1. INTRODUCTION

In [2] Bleimann, Butzer and Hahn have introduced a new Bernstein type operator L_n which is given on the space of continuous functions on the unbounded interval $[0, \infty)$. This positive linear operator is defined by

$$(1) \qquad (L_n f)(x) = (1+x)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \qquad n \in \mathbb{N}.$$

The authors proved that for $f \in C[0, \infty)$, $L_n f \to f$ as $n \to \infty$ pointwise on $[0, \infty)$, the convergence being uniform on each compact subset of $[0, \infty)$. Furthermore, estimations for the rate of convergence of $|(L_n f)(x) - f(x)|$ were established in terms of the second modulus of continuity of f, where f is assumed to be bounded and uniformly continuous on $[0, \infty)$. This operator has the virtue of being a finite sum which arises in a natural way and not as a truncation process of an operator defined by means of an infinite sum. La-

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tely, several authors have studied the operator L_n, for example U. Abel [1], Jesús de la Cal and F. Luquin [3], R.A.Khan [4], A.Mercer [6].

We mention that the demonstration of the Voronowskaya type theorem given by Mercer is based on the observation that these operators are intimately related to the Bernstein operators. By the rational transformation $y = x/(1+x)$ the operators become

$$(2) \qquad \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} F\left(\frac{k}{n+1}\right), \qquad n \in \mathbb{N},$$

with $F(y) = f(y/(1-y))$.

In [7] D.D.Stancu considered the polynomials of Bernstein type:

$$(S_n^{a,b} g)(x) = \sum_{k=0}^{n} p_{n,k}(x) g\left(\frac{k+a}{n+b}\right), \qquad n \in \mathbb{N},$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad g \in C[0,1]$$

and a, b are real parameters, independently of n, such that $0 \le a \le b$. It is clear that for $g = F$, $a = 0$ and $b = 1$ we obtain (2).

The purpose of this paper is to present a Bleimann, Butzer and Hahn operator with slightly modified nodes. The initial operator is characterized by the fact that it uses the spaced nodes $x_k = \dfrac{k}{n-k+1}$, $k = 0, 1, \ldots, n$; now we take the nodes $x_k^a = \dfrac{k+a}{n-k+1}$ where a is a real and positive parameter. In this way we obtain the following class of operators:

$$(3) \qquad (L_n^a f)(x) = (1+x)^{-n} \sum_{k=0}^{n} f\left(\frac{k+a}{n-k+1}\right) \binom{n}{k} x^k, \qquad n \in \mathbb{N}.$$

It is evident that these operators are linear, positive and for $a = 0$, $L_n^0 \equiv L_n$. Also they are bounded in the sense that for $x \ge 0$, $n \in \mathbb{N}$, one has

$$(4) \qquad |(L_n^a f)(x)| \le \|f\|_{C_B}, \qquad f \in C_B[0, \infty)$$

since

$$(5) \qquad (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k = 1,$$

$C_B[0,\infty)$ being the class of real-valued functions f defined on $[0,\infty)$ which are bounded and uniformly continuous with norm

$$\|f\|_{C_B} = \sup_{t\geq 0} |f(t)|.$$

2. AUXILIARY RESULTS

Before establishing the main results we present some identities and inequalities which will be used later.

Lemma 1. *For the operators $L_n^a : C[0,\infty) \to C[0,\infty)$ defined at (3) the next relations*

(i) $(L_n^a e_0)(x) = 1$,

(ii) $(L_n^a e_1)(x) = x + a\dfrac{x+1}{n+1} - \left(\dfrac{a}{n+1}+1\right)x\left(\dfrac{x}{1+x}\right)^n$,

(iii) $(L_n^a e_2)(x) \leq (L_n^0 e_2)(x) + \dfrac{a}{n+1}(x+1)(4x+a) - a\left(\dfrac{4x+a}{n+1}+4\right)x\left(\dfrac{x}{x+1}\right)^n$

hold, where $e_j(t) = t^j$ $(j = 0,1,2)$ for any $t \in [0,\infty)$.

Proof. The first identity follows immediately from the relation (5). For the second identity we shall use the simple identity:

$$\frac{1}{n-k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k}.$$

We can write successively:

$$(6)\quad \begin{aligned} (L_n^a e_1)(x) &= (L_n^0 e_1)(x) + \frac{a}{(1+x)^n}\sum_{k=0}^{n}\frac{1}{n-k+1}\binom{n}{k}x^k = \\ &= (L_n^0 e_1)(x) + \frac{a}{(n+1)(1+x)^n}\sum_{k=0}^{n}\binom{n+1}{k}x^k = \\ &= (L_n^0 e_1)(x) + \frac{a}{(n+1)(1+x)^n}((1+x)^{n+1} - x^{n+1}). \end{aligned}$$

From [2] (see Lemma 1) there follows:

$$(L_n^0 e_1)(x) = x - x\left(\frac{x}{1+x}\right)^n.$$

Taking this relation into account, (6) leads us to the desired result. For the monomial e_2 we can write:

$$(L_n^a e_2)(x) = (L_n^0 e_2)(x) + 2a\frac{1}{(1+x)^n}\sum_{k=0}^{n}\frac{k}{(n-k+1)^2}\binom{n}{k}x^k +$$

$$(7) \qquad +a^2\frac{1}{(1+x)^n}\sum_{k=0}^{n}\frac{1}{(n-k+1)^2}\binom{n}{k}x^k.$$

Because

$$\frac{k}{(n-k+1)^2}\binom{n}{k} \leq \frac{2}{n+1}\binom{n+1}{k-1}, \qquad k = 1, 2, \ldots, n$$

from the first sum of the relation (7) we obtain:

$$\frac{1}{(1+x)^n}\sum_{k=1}^{n}\frac{k}{(n-k+1)^2}\binom{n}{k}x^k \leq \frac{1}{(1+x)^n}\frac{2}{n+1}\sum_{k=1}^{n}\binom{n+1}{k-1}x^k =$$

$$= \frac{1}{(1+x)^n}\frac{2x}{n+1}\left\{(1+x)^{n+1} - \binom{n+1}{n}x^n - x^{n+1}\right\} =$$

$$(8) \qquad = \frac{2x(x+1)}{n+1} - 2x\left(\frac{x}{x+1}\right)^n - \frac{2x^2}{n+1}\left(\frac{x}{x+1}\right)^n.$$

In a similar way, starting from the inequality

$$\frac{1}{(n-k+1)^2}\binom{n}{k} \leq \frac{1}{n+1}\binom{n+1}{k}, \qquad k = 0, 1, \ldots, n$$

the second sum of the relation (7) becomes:

$$\frac{1}{(1+x)^n}\sum_{k=0}^{n}\frac{1}{(n-k+1)^2}\binom{n}{k}x^k \leq \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\frac{x^k}{(1+x)^n} =$$

$$(9) \qquad = \frac{1}{n+1}\frac{(1+x)^{n+1} - x^{n+1}}{(1+x)^n} =$$

$$= \frac{x+1}{n+1} - \frac{x}{n+1}\left(\frac{x}{x+1}\right)^n.$$

If we insert (8) and (9) into the relation (7) we obtain the desired conclusion, after a few calculations.

Lemma 2. *Let be the operators $L_n^a : C[0,\infty) \to C[0,\infty)$ defined at (3). There hold for $x \in [0,\infty)$:*

(i) $|(L_n^a e_1)(x) - x| \le \frac{x+1}{n+1}\left\{x+1+a\left(1+\frac{x+1}{n+1}\right)\right\}$, $n \in \mathbb{N}$,

(ii) $|L_n^a((e_1(t)-x)^2;x)| \le \frac{x+1}{n+1}\left\{a^2+2x\left(1+\frac{x+1}{n+1}\right)a+4x(x+1)\right\}$, $n \ge 24(1+x)$.

Proof. In order to demonstrate this lemma we remind the Bleimann, Butzer and Hahn's formula [2]:

$$|(L_n^0 e_2)(x) - x^2| \le \frac{2x(1+x)^2}{n+2}, \quad x \ge 0, \quad n \ge 24(1+x). \tag{10}$$

Also, we need the inequality

$$x\left(\frac{x}{x+1}\right)^n \le \frac{(x+1)^2}{n+1}, \quad x \ge 0, \quad n \in \mathbb{N}, \tag{11}$$

which evidently derives from $(n+1)x^{n+1} \le (n+2)x^{n+1} \le (1+x)^{n+2}$. Starting from Lemma 1 and taking into account (11), we can write:

$$|(L_n^a e_1)(x) - x| \le \frac{a(x+1)}{n+1} + \left(\frac{a}{n+1}+1\right)x\left(\frac{x}{x+1}\right)^n \le$$

$$\le \frac{x+1}{n+1}\left\{x+1+a\left(1+\frac{x+1}{n+1}\right)\right\}.$$

Thus (i) has been demonstrated. Using the same lemma one observes that

$$L_n^a((e_1(t)-x)^2;x) = (L_n^a e_2)(x) - 2x(L_n^a e_1)(x) + x^2(L_n^a e_0)(x) \le$$

$$\le (L_n^0 e_2)(x) - x^2 + \frac{x+1}{n+1}\left\{a^2+2x\left(1+\frac{x+1}{n+1}\right)a+2x(x+1)\right\}.$$

The above inequality and the relation (10) imply the aimed conclusion.

3. MAIN RESULTS

Theorem 1. *If* $f \in C[0,\infty)$, *then*

$$\lim_{n\to\infty}(L_n^a f)(x) = f(x)$$

at each point $x \in [0,\infty)$, *the convergence being uniform on each compact subinterval* $[0,b]$ *of* $[0,\infty)$.

Proof. To establish this result, it suffices to apply the pointwise version of the well-known theorem of Bohman-Korovkin [5], noting that $(L_n e_i)(x) \to$ $\to e_i(x)$ for $i = 0,1,2$ and each $x \geq 0$ derive from Lemma 1.

Theorem 2. *If* $f \in C_B^2[0,\infty)$, *there holds for all* $n \geq 24(1+x)$

$$|(L_n^a f)(x) - f(x)| \leq$$

$$\leq \frac{(x+1)^2}{n+1}\left\{\frac{a^2}{2(x+1)} + \left(1+\frac{x+1}{n+1}\right)a + 2x + 1\right\}\|f\|_{C_B^2}. \tag{12}$$

Proof. To begin, we notice that $C_B^2[0,\infty) = \{f \in C_B[0,\infty); f', f'' \in$ $\in C_B[0,\infty)\}$ and the norm is given by $\|f\|_{C_B^2} = \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}$. We apply the Taylor expansion to $f \in C_B^2[0,\infty)$,

$$\begin{aligned}(L_n^a f)(x) - f(x) &= L_n^a(f(t) - f(x); x) = \\ &= f'(x)L_n^a(e_1(t) - x; x) + \tfrac{1}{2}L_n^a((e_1(t)-x)^2 f''(\xi); x),\end{aligned}$$

where ξ lies between t and x. The positivity of L_n^a together with Lemma 2 now imply that

$$\begin{aligned}|(L_n^a f)(x) - f(x)| &\leq |(L_n^a e_1)(x) - x|\|f'\|_{C_B} + \frac{1}{2}|L_n((e_1(t)-x)^2; x)|\|f''\|_{C_B} \leq \\ &\leq \frac{x+1}{n+1}\left\{x+1+a\left(1+\frac{x+1}{n+1}\right)\right\}\|f'\|_{C_B} + \\ &\quad + \frac{x+1}{n+1}\left\{\frac{a^2}{2} + x\left(1+\frac{x+1}{n+1}\right)a + 2x(x+1)\right\}\|f''\|_{C_B} \leq \\ &\leq \frac{x+1}{n+1}\left\{\frac{a^2}{2} + (x+1)\left(1+\frac{x+1}{n+1}\right)a + (x+1)(2x+1)\right\}(\|f'\|_{C_B} + \|f''\|_{C_B}).\end{aligned}$$

Consequently, it will result (12).

An instrument required is the K-functional of $f \in C_B[0,\infty)$, defined for $t \geq 0$ by

$$K(t,f) = \inf_{g \in C_B^2} \left\{ \|f-g\|_{C_B} + t\|g\|_{C_B^2} \right\}.$$

We can state and prove the following result:

Corollary 1. *If* $f \in C_B[0,\infty)$, $x \in [0,\infty)$, $n \geq 24(1+x)$, *there holds*

$$|(L_n^a f)(x) - f(x)| \leq 2K(A(n,a,x),f) \tag{13}$$

where

$$A(n,a,x) = \frac{(x+1)^2}{2(n+1)} \left\{ \frac{a^2}{2(x+1)} + \left(1 + \frac{x+1}{n+1}\right) a + 2x + 1 \right\}.$$

Proof. Once inequality (12) is known, the proof follows standard means. For $f \in C_B[0,\infty)$, $g \in C_B^2[0,\infty)$, $n \geq 24(1+x)$ by (4) one has:

$$|(L_n^a f)(x) - f(x)| \leq |(L_n^a f)(x) - (L_n^a g)(x)| + |(L_n^a g)(x) - g(x)| + |g(x) - f(x)| \leq$$

$$\leq 2\|f-g\|_{C_B} + \frac{(x+1)^2}{n+1} \left\{ \frac{a^2}{2(x+1)} + \left(1 + \frac{x+1}{n+1}\right) a + 2x + 1 \right\} \|g\|_{C_B^2}.$$

Since the left side is independent of g, applying the definition of the K-functional we obtain the relation (13).

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