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ON THE MONOTONICITY OF A SEQUENCE OF STANCU-BERNSTEIN TYPE OPERATORS

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Dedicated to Prof. Gheorghe Coman at his 60^{th} anniversary

Abstract. One makes a study of a sequence of Bernstein type operators, introduced and studied in [9]. These are depending on two parameters a and b, $0 \leq a \leq b$. First, one deduces a representations by divided differences for the difference of two consecutive terms of the sequence of polynomials obtained by appliing these operators to a function $f \in C[0,1]$. Using this representation, one enounces several sufficient conditions for the monotony of the sequence of Stancu-Bernstein polynomials.

1. Introduction

In 1969 D.D.Stancu [9] considered and studied the following generalization of the Bernstein polynomial:

$$\left(S_n^{a,b} f\right)(x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k+a}{n+b}\right), \tag{1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{2}$$

and a,b are real parameters, independent of n, such that $0 \leq a \leq b$. This is an interpolatory type polynomial characterized by the fact that it uses equally spaced nodes $x_k = \frac{k+a}{n+b}$ $(k = 0,1,\ldots,n)$. If $ab \neq 0$ and $a \neq b$ then it does not coincide at any node with the function f; if $a = 0$ and $b \neq 0$ then it coincides with f at $x_0 = 0$, while if $a = b \neq 0$ then it coincides with f at $x_n = 1$. When $a = b = 0$ one obtains the classical Bernstein polynomial.

It was proved that for $f \in C[0,1]$ the sequence of the polynomials (1) converges uniformly to f on [0,1]. Then the corresponding order of approximation was evaluated by using the modulus of continuity of f; also there were deduced expressions for the

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remainder term of the approximation formula

$$f(x) = \left(S_n^{a,b} f\right)(x) + \left(R_n^{a,b} f\right)(x)$$

and, finally, the author presented a theorem of Voronovskaja type giving an asymptotic estimation for the remainder term. In this paper we shall investigate the monotonicity properties of this sequence.

2. The basic theorem

In order to study the monotonicity of the sequence $\left(S_n^{a,b} f\right)$ we shall establish a useful formula for the difference of two consecutive terms of the Stancu-Bernstein polynomials. The following theorem holds:

Theorem 1. *The difference between the polynomials* $\left(S_{n+1}^{a,b} f\right)(x)$ *and* $\left(S_n^{a,b} f\right)(x)$ *can be expressed under the form:*

$$\left(S_{n+1}^{a,b} f\right)(x) - \left(S_n^{a,b} f\right)(x) = -\frac{nx(1-x)}{(n+b)(n+b+1)} \sum_{k=0}^{n-1} \binom{n}{k} \cdot$$
$$\cdot \left(\frac{1}{n+b} \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f \right] + \frac{a}{k+1} \left[\frac{k+1+a}{n+1+b}, \frac{k+1+a}{n+b}; f \right], \right.$$
$$\left. - \frac{b-a}{n-k} \left[\frac{k+a}{n+b}, \frac{k+1+a}{n+1+b}; f \right] \right) x^k (1-x)^{n-k-1} + \left(U_n^{a,b} f\right)(x), \tag{3}$$

where the brackets represent the symbol for divided differences and

$$\left(U_n^{a,b} f\right)(x) = \left(f\left(\frac{a}{n+1+b}\right) - f\left(\frac{a}{n+b}\right) \right) (1-x)^{n+1} +$$
$$+ \left(f\left(\frac{n+1+a}{n+1+b}\right) - f\left(\frac{n+a}{n+b}\right) \right) x^{n+1}. \tag{4}$$

Proof. First we write:

$$\left(S_{n+1}^{a,b} f\right)(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k (1-x)^{n+1-k} f\left(\frac{k+a}{n+1+b}\right) =$$
$$= \sum_{k=1}^{n} \binom{n+1}{k} x^k (1-x)^{n+1-k} f\left(\frac{k+a}{n+1+b}\right) +$$
$$+ (1-x)^{n+1} f\left(\frac{a}{n+1+b}\right) + x^{n+1} f\left(\frac{n+1+a}{n+1+b}\right) \tag{5}$$

Then let us consider the relation:

$$\left(S_n^{a,b}f\right)(x)=\sum_{k=0}^{n}xP_{n,k}(x)f\left(\frac{k+a}{n+b}\right)+\sum_{k=0}^{n}(1-x)P_{n,k}(x)f\left(\frac{k+a}{n+b}\right)$$

In the first sum from the right-hand side of this equality we set $i=k+1$ and then denote again the summation index by k we obtain:

$$\begin{aligned}\left(S_n^{a,b}f\right)(x) &= \sum_{k=1}^{n+1}xP_{n,k-1}(x)f\left(\frac{k-1+a}{n+b}\right)+\sum_{k=0}^{n}(1-x)P_{n,k}(x)f\left(\frac{k+a}{n+b}\right)=\\ &= \sum_{k=1}^{n}\binom{n}{k-1}x^k(1-x)^{n-k+1}f\left(\frac{k-1+a}{n+b}\right)+x^{n+1}f\left(\frac{n+a}{n+b}\right)+\\ &+ (1-x)^{n+1}f\left(\frac{a}{n+b}\right)+\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{n+1-k}f\left(\frac{k+a}{n+b}\right)\end{aligned}\tag{6}$$

By using (5) and (6) we get:

$$\begin{aligned}\left(S_{n+1}^{a,b}f\right)(x)-\left(S_n^{a,b}f\right)(x) &= \sum_{k=1}^{n}\left(\binom{n+1}{k}f\left(\frac{k+a}{n+1+b}\right)-\right.\\ &\left.-\binom{n}{k-1}f\left(\frac{k-1+a}{n+b}\right)-\binom{n}{k}f\left(\frac{k+a}{n+b}\right)\right)x^k(1-x)^{n-k+1}\\ &+\left(f\left(\frac{a}{n+1+b}\right)-f\left(\frac{a}{n+b}\right)\right)(1-x)^{n+1}+\\ &+\left(f\left(\frac{n+1+a}{n+1+b}\right)-f\left(\frac{n+a}{n+b}\right)\right)x^{n+1}\end{aligned}$$

If we use the identities:

$$\binom{n}{k-1}=\frac{k}{n+1-k}\binom{n}{k},\quad \binom{n+1}{k}=\frac{n+1}{n+1-k}\binom{n}{k}$$

and the notation defined at (4), we can write:

$$\begin{aligned}\left(S_{n+1}^{a,b}f\right)(x)-\left(S_n^{a,b}f\right)(x) &= -\sum_{k=1}^{n}\binom{n}{k}\left(f\left(\frac{k+a}{n+b}\right)+\frac{k}{n-k+1}f\left(\frac{k-1+a}{n+b}\right)-\right.\\ &\left.-\frac{n+1}{n-k+1}f\left(\frac{k+a}{n+1+b}\right)\right)x^k(1-x)^{n-k+1}+\left(U_n^{a,b}f\right)(x)\end{aligned}$$

If we make the change $k = j+1$ and then denote again the summation index by k we have:

$$\left(S_{n+1}^{a,b}f\right)(x)-\left(S_n^{a,b}f\right)(x)=$$

$$=-\sum_{k=0}^{n-1}\binom{n}{k+1}\left(f\left(\frac{k+1+a}{n+b}\right)+\frac{k+1}{n-k}f\left(\frac{k+a}{n+b}\right)-\right.$$

$$\left.-\frac{n+1}{n-k}f\left(\frac{k+1+a}{n+1+b}\right)\right)x^{k+1}(1-x)^{n-k}+$$

$$+\left(U_n^{a,b}f\right)(x)=-x(1-x)\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{n}{k+1}f\left(\frac{k+1+a}{n+b}\right)+\frac{n}{n-k}f\left(\frac{k+a}{n+b}\right)-\right.$$

$$\left.\frac{n(n+1)}{(k+1)(n-k)}f\left(\frac{k+1+a}{n+1+b}\right)\right)x^k(1-x)^{n-k-1}+\left(U_n^{a,b}f\right)(x) \tag{7}$$

since

$$\binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k}.$$

According to the fact that:

$$\frac{n(n+1)}{(k+1)(n-k)}f\left(\frac{k+1+a}{n+1+b}\right)=n\left(\frac{1}{k+1}+\frac{1}{n-k}\right)f\left(\frac{k+1+a}{n+1+b}\right),$$

we can write successively:

$$\frac{n}{k+1}f\left(\frac{k+1+a}{n+b}\right)+\frac{n}{n-k}f\left(\frac{k+a}{n+b}\right)-\frac{n(n+1)}{(k+1)(n-k)}f\left(\frac{k+1+a}{n+1+b}\right)=$$

$$=\frac{n}{k+1}\left(f\left(\frac{k+1+a}{n+b}\right)-f\left(\frac{k+1+a}{n+1+b}\right)\right)+\frac{n}{n-k}\left(f\left(\frac{k+a}{n+b}\right)-f\left(\frac{k+1+a}{n+1+b}\right)\right)=$$

$$=\frac{n}{(n+b)(n+b+1)}\left\{\frac{k+1+a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\right.$$

$$\left.-\frac{(n-k)+(b-a)}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\right\}=$$

$$=\frac{n}{(n+b)(n+b+1)}\left\{\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]+\right.$$

$$+\frac{a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\frac{b-a}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\Bigg\}=$$

$$=\frac{n}{(n+1)(n+b+1)}\left\{\frac{1}{n+b}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]+\right.$$

$$\left.+\frac{a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\frac{b-a}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\right\} \tag{8}$$

Taking (8) into account, the equality (7) leads us to the desired formula (3).

□

We notice that if $a=b=0$ our result becomes:

$$\left(S_{n+1}^{0,0}f\right)(x)-\left(S_{n}^{0,0}f\right)(x)=$$

$$=-\frac{x(1-x)}{n(n+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]x^k(1-x)^{n-k+1}=$$

$$=-\frac{x(1-x)}{n(n+1)}\sum_{k=0}^{n-1}P_{n-1,k}(x)\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]$$

This formula was established by D.D.Stancu in the paper [8].

3. Sufficient conditions to ensure the monotonicity of sequence

The following definition of the notion of higher-order convex functions is known (see [7]).

Definition 1. A real-valued function on an interval I is called convex of order n on I if all its divided differences of order $n+1$, on $n+2$ distinct points of I, are positive. The function f is said to be non-concave of order n on the interval I is all its divided differences of order $n+1$, on any $n+2$ points of I, are non-negative.

We shall consider the next particular cases:

A. If we choose $a=b>0$ in Theorem 1 we obtain

$$\left(S_{n+1}^{a,a}f\right)(x)-\left(S_{n}^{a,a}f\right)(x)=$$

$$=-\frac{nx(1-x)}{(n+a)(n+a+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left\{\frac{1}{n+a}\left[\frac{k+a}{n+a},\frac{k+a+1}{n+a+1},\frac{k+a+1}{n+a};f\right]+\right.$$

$$+\frac{a}{k+1}\left[\frac{k+a+1}{n+a+1},\frac{k+a+1}{n+a};f\right]\Big\} x^k(1-x)^{n-k-1} -$$

$$-\frac{a}{(n+a)(n+a+1)}\left[\frac{a}{n+a+1},\frac{a}{n+a};f\right](1-x)^{n+1}.$$

Since on the interval [0,1] we have $P_{n-1,k}(x) \geq 0$ $(k = 0,1,...,n-1)$, from this identity and Definition 1 there follows:

Theorem 2. *If the function f is convex of first order on the interval [0,1] and increasing on [0,1] then the sequence* $(S_n^{a,b}f)$ *is decreasing on (0,1), that is* $(S_n^{a,a}f) > (S_{n+1}^{a,a}f)$ *on (0,1) for n = 1,2,...*

Analysing similarly it is easy to state:

Corollary 1. *If the function f is concave of first order on the interval [0,1] and decreasing on [0,1] then the sequence* $(S_n^{a,a}f)$ *is increasing on the interval (0,1).*

B. If we take $a = 0$ and $b > 0$ in Theorem 1 we have:

$$\left(S_{n+1}^{0,b}f\right)(x) - \left(S_n^{0,b}f\right)(x) =$$

$$= -\frac{nx(1-x)}{(n+b)(n+b+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\Big\{\frac{1}{n+b}\left[\frac{k}{n+b},\frac{k+1}{n+b+1},\frac{k+1}{n+b};f\right] -$$

$$-\frac{b}{n-k}\left[\frac{k}{n+b},\frac{k+1}{n+b+1};f\right]\Big\} x^k(1-x)^{n-k-1} + \frac{b}{(n+b)(n+b+1}\left[\frac{n+1}{n+b+1},\frac{n}{n+b};f\right]$$

Now we can state the following proposition:

Theorem 3. *If the function f is convex of first order on the interval [0,1] and decresing on [0,1] then sequence* $(S_n^{0,b}f)$ *is decreasing on the interval (0,1), that is*

$$\left(S_n^{0,b}f\right) > \left(S_{n+1}^{0,b}f\right) \tag{9}$$

on (0,1) and n = 1,2,...

Analogously we formulate:

Corollary 2. *If the function f is concave of first order on the interval [0,1] and increasing on [0,1] then the sequence* $\left(S_n^{0,b}f\right)$ *is decreasing on the interval (0,1).*

To conclude, we mention that different liniar approximation operators introduced and studied by D.D.Stancu (mainly by probabilistic methods) have been the object of other investigations, made by many other researchers - see [1], [2], [3], [4], [5], [6].

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