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APPROXIMATION THEOREM IN L_p FOR A CLASS OF OPERATORS CONSTRUCTED BY WAVELETS

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Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. In this paper we deal with a linear operator of Baskakov-type which has been constructed in [1] by using wavelets. Now, we estimate the order of approximation in L_p-spaces ($1 < p \leq \infty$) for smooth functions.

1. Introduction

Recently, "wavelets" have become a versatile tool in both theoretical and applied mathematics. Certain families of functions generated by dilations and translations of a single function ψ, i.e. given by

$$\psi_{a,b}(x) = |a|^{-1/2}\psi(ax-b), \quad a, b \in \mathbf{R},\ a \neq 0,$$

have been studied in many works, see [3], [4], [6]. By using wavelets it is possible to construct classes of operators which are useful in the approximation theory. In [5] H.H. Gonska and Ding-Xuan Zhou introduced a class of Szász-type operators by means of Daubechies'compactly supported wavelets, which have the advantage that they can be used for L_p-approximation ($1 < p \leq \infty$). Following the same idea, in [1] was presented Baskakov-type operators. These operators are defined as

$$(L_n f)(x) = n\sum_{k=0}^{\infty} b_{n,k}(x)\int_{\mathbf{R}} f(t)\psi(nt-k)dt = \sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\lambda} f\left(\frac{t+k}{n}\right)\psi(t)dt, \qquad (1)$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0,$$

and ψ belongs to $L_\infty(\mathbf{R})$ such as supp$\psi \subset [0, \lambda]$ with $0 < \lambda < \infty$. Also, for ψ we require the following conditions:

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(i) its first r moments vanish:

$$\int_{\mathbf{R}} t^k \psi(t)dt = 0, \quad 1 \le k \le r, \tag{2}$$

and

(ii)

$$\int_{\mathbf{R}} \psi(t)dt = 1. \tag{3}$$

The condition (i) implies that our operators have the same moments as Baskakov operators in an arbitrarily chosen number. When $\psi = \chi_{[0,1)}$ L_n are exactly the Baskakov-Kantorovich operators B_n^* given by:

$$(B_n^* f)(x) = n \sum_{k=0}^{\infty} b_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du.$$

The main goal in this paper is to give an approximation theorem in L_p $(1 < p \le \infty)$ for the operators introduced by (1).

2. Results

Firstly, we recall the Hardy-Littlewood maximal function M of a locally integrable function g, that is $g \in L_1^{loc}$. It is a sublinear operator of the kind

$$(Mg)(x) = \sup_{t \ne x} \left| \frac{1}{t-x} \int_x^t |g(u)|du \right|. \tag{4}$$

Obviously $\|Mg\|_\infty \le \|g\|_\infty$. Further, an application of Marcinkiewicz's theorem (see [2], page 80) leads to the relation

$$\|Mg\|_p \le \gamma(p) \|g\|_p \quad 1 < p \le \infty, \tag{5}$$

where $\gamma(p)$ is a constant and $\|\cdot\|_p$ indicates the norm of the Banach space L_p. In our investigation we shall use the function φ,

$$\varphi(x) = \sqrt{x(x+1)}, \quad x \ge 0, \tag{6}$$

which represents the step-weight function related to the operators of Baskakov and Baskakov Kantorovich.

Also, we need the first moments of Baskakov operators:

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = 0, \quad \mu_{n,2}(x) = \frac{\varphi^2(x)}{n}. \tag{7}$$

where, generally

$$\mu_{n,m}(x) = \sum_{k=0}^{\infty} b_{n,k}(x) \left(\frac{k}{n} - x\right)^m, \quad m \in \mathbb{N}.$$

Finally, we note with $A.C._+^{loc}$ the space of all functions absolutely continuous in every closed finite and positive interval.

Now, we mention some results in the form of lemmas which will be used in the sequel.

Lemma 1. *If f and f' belong to $A.C._+^{loc}$ then, for any $x \in \left[0, \frac{1}{\sqrt{n}}\right]$, the following inequality*

$$|(L_n f)(x) - f(x)| \leq \frac{\lambda(\lambda + 3\sqrt{n})}{2n} \|\psi\|_\infty (Mf')(x)$$

holds.

Proof. Starting from the identity $f(t) = f(x) + \int_x^t f'(u)du$, and using the relations (1) and (4) we can write successively:

$$|(L_n f)(x) - f(x)| = \left| L_n \left(\int_x^t f'(u)du; x \right) \right| \leq \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\lambda \left| \int_x^{\frac{t+k}{n}} |f'(u)| du \right| dt \|\psi\|_\infty \leq$$

$$\leq \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\lambda \left| \frac{t+k}{n} - x \right| dt \|\psi\|_\infty (Mf')(x). \tag{8}$$

Further, we have:

$$\int_0^\lambda \left| \frac{t+k}{n} - x \right| dt \leq \frac{\lambda^2}{2n} + \lambda \left| \frac{k}{n} - x \right|. \tag{9}$$

Applying Cauchy's inequality and taking into account the relations (7), we obtain for any $x \in \left[0, \frac{1}{\sqrt{n}}\right]$:

$$\sum_{k=0}^{\infty} b_{n,k}(x) \left| \frac{k}{n} - x \right| \leq \mu_{n,2}^{1/2}(x) < \frac{3}{2\sqrt{n}}. \tag{10}$$

Substituting (9) and (10) in relation (8) we arrive at the desired result. □

Lemma 2. *Let $x > \frac{1}{\sqrt{n}}$. If we define:*

$$A_n(x, \lambda) = \frac{1}{x(x+1)} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\lambda \left(x - \frac{t+k}{n} \right)^2 dt,$$

$$B_n(x, \lambda) = \frac{1}{x} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\lambda \frac{\left(x - \frac{t+k}{n}\right)^2}{1 + \frac{t+k}{n}} dt,$$

then

$$A_n(x,\lambda) < \left(\frac{\lambda^3}{3}+\lambda\right)\frac{1}{n}, \quad n \geq 1, \tag{11}$$

and

$$B_n(x,\lambda) < \left(\frac{\lambda^2}{2}+4\lambda\right)\frac{1}{\sqrt{n}}, \quad n \geq 2. \tag{12}$$

Proof. According to (7) we have:

$$A_n(x,\lambda) = \frac{1}{x(x+1)}\int_0^\lambda \left(\mu_{n,2}(x) - \frac{2}{n}t\mu_{n,1}(x) + \frac{t^2}{n^2}\right)dt = \frac{\lambda}{n} + \frac{\lambda^3}{3n^2x(x+1)}.$$

The assumption $\sqrt{n}x > 1$ implies the relation (11).

To estimate $B_n(x,\lambda)$ we notice that

$$\begin{aligned} B_n(x,\lambda) &= \frac{n}{x}\sum_{k=0}^\infty b_{n,k}(x)\int_{1+\frac{k}{n}}^{1+\frac{\lambda+k}{n}} \frac{(x+1-y)^2}{y}dy = \\ &= \frac{n(x+1)^2}{x}\sum_{k=0}^\infty b_{n,k}(x)\ln\left(1+\frac{\lambda}{n+k}\right) + \frac{\lambda^2}{2nx} - \lambda\frac{x+1}{x}. \end{aligned} \tag{13}$$

From the inequality $\ln(1+t) < t$, $t > 0$, and (7) it follows for every $n \geq 2$ that:

$$\begin{aligned} n\sum_{k=0}^\infty b_{n,k}(x)\ln\left(1+\frac{\lambda}{n+k}\right) &< \frac{\lambda}{1+x}\sum_{k=0}^\infty \frac{(n+k-2)!}{k!(n-2)!}\frac{x^k}{(1+x)^{n+k-1}}\frac{n+k-1}{n+k}\frac{n}{n-1} < \\ &< \frac{\lambda}{1+x}\mu_{n-1,0}(x)\frac{n}{n-1} = \frac{\lambda}{1+x}\frac{n}{n-1}. \end{aligned}$$

Returning to (13) we deduce:

$$B_n(x,\lambda) < \frac{\lambda^2}{2nx} + \lambda\left(1+\frac{1}{x}\right)\frac{1}{n-1} < \frac{\lambda(\lambda+8)}{2\sqrt{n}},$$

because $\frac{1}{x} < \sqrt{n}$. Hence (12) holds. □

Lemma 3. *If f and $\varphi^2 f''$ belong to $A.C._+^{loc}$ then, for any $x > \frac{1}{\sqrt{n}}$ and $n \geq 2$, the following inequality*

$$|(L_nf)(x) - f(x)| \leq \left(\frac{\lambda^3}{3}+\frac{\lambda^2}{2}+5\lambda\right)\frac{1}{\sqrt{n}}\|\psi\|_\infty M(\varphi^2 f'')(x)$$

holds.

Proof. We use the Taylor formula

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du$$

and considering (1) we obtain:

$$(L_n f)(x) - f(x) = f'(x) L_n(\tau_x; x) + L_n(R_{2,x} f; x),$$

where $\tau_x(t) = t - x$ and

$$(R_{2,x} f)(t) = \int_x^t (t-u) f''(u) du.$$

The conditions (2) and (3) ensure $L_n(\tau_x; x) = 0$, consequently we get:

$$|(L_n f)(x) - f(x)| \leq \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\lambda} \left| \int_x^{\frac{t+k}{n}} \left| \frac{t+k}{n} - u \right| |f''(u)| du \right| dt \|\psi\|_\infty. \quad (14)$$

We use the fact that:

$$\frac{|v-u|}{u(1+u)} \leq \frac{|x-v|}{x(1+u)} \leq \frac{|x-v|}{x} \left(\frac{1}{1+x} + \frac{1}{1+v} \right)$$

for every u between x and v $(x > 0,\ v > 0)$, and choosing $v = \frac{t+k}{n}$ we can write:

$$\left| \frac{t+k}{n} - u \right| |f''(u)| \leq \frac{\left| x - \frac{t+k}{n} \right|}{x} \left(\frac{1}{1+x} + \frac{1}{1+\frac{k+t}{n}} \right) |\varphi^2(u) f''(u)|,$$

where φ is defined in (6).

We place this above inequality in (14) and taking into account both the definition of Hardy-Littlewood operator from (4) and the notations which were introduced in Lemma 2, we have:

$$|(L_n f)(x) - f(x)| \leq (A_n(x,\lambda) + B_n(x,\lambda)) \|\psi\|_\infty M(\varphi^2 f'')(x).$$

Recalling now (11) and (12) the proof of Lemma 3 is complete. □

Combining the cases of Lemma 1 and Lemma 3 after a new increase, we have for $x \in [0,\infty)$ and $n \geq 2$:

$$|(L_n f)(x) - f(x)| \leq \left(\frac{\lambda^3}{3} + \lambda^2 + \frac{13}{2}\lambda \right) \frac{\|\psi\|_\infty}{\sqrt{n}} ((Mf')(x) + M(\varphi^2 f'')(x)).$$

This implies for $1 < p \leq \infty$ and $f', \varphi^2 f'' \in L_p[0,\infty)$:

$$\|L_n f - f\|_p \leq \frac{c_\lambda \|\psi\|_\infty}{\sqrt{n}} (\|M(f')\|_p + \|M(\varphi^2 f'')\|_p),$$

where c_λ is a constant.

If we use the relation (4) in this above inequality, we are able to state our main result.

Theorem. *Let $1 < p \leq \infty$ and L_n the operators defined in (1). If f, f' and $\varphi^2 f''$ belong to $A.C.^{loc}_{+} \cap L_p[0,\infty)$, then we have*

$$\|L_n f - f\|_p \leq \frac{M}{\sqrt{n}}(\|f'\|_p + \|\varphi^2 f''\|_p)\|\psi\|_\infty,$$

where M is a constant which depends on λ and p.

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