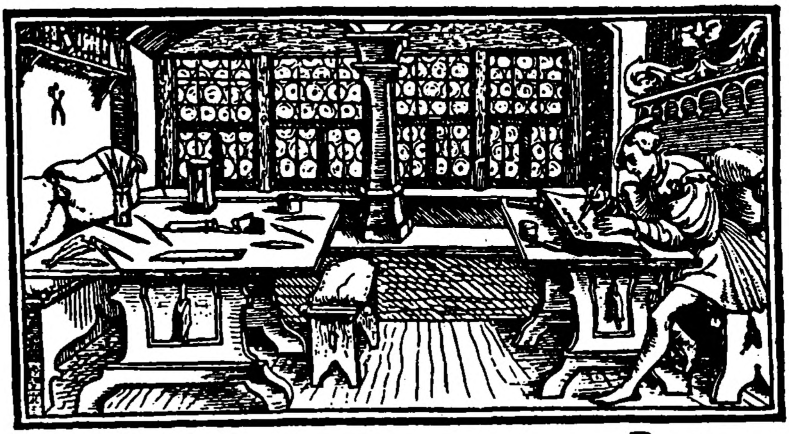
STUDIA
UNIVERSITATIS BABES-BOLYAI

Mathematica

CLUJ-NAPOCA 1997



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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLII, Number 4, December 1997

ON A FUNCTIONAL EQUATION

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Abstract. This is a survey paper devoted to the following functional equation

$$\sum_{k\in\mathbf{Z}} p_k u(x-k) = v(x), \quad x \in \mathbf{R},$$

which is in connection with the notion of wavelets. If $v(k)$ vanishes for $k \in \mathbf{Z}$ and if $p_k = 0$ for $k < 0$ and $k \geq m+1$, then, for $x = n$, the above equation leads us to the well-known general m^{th}-order linear recurrence relation. For $v(x) = u(2x)$, $x \in \mathbf{R}$, we present how this equation appears as a necessity in the field of mathematics. We also indicate three properties which must be fulfilled by the function and the sequence so that these equations admit solutions. When the sequence $(p_k)_{k\in\mathbf{Z}}$ has a compact support other properties are revealed and the technique to obtain solutions is described.

1. Introduction

Starting from the general m^{th}-order linear recurrence relation

$$\sum_{k=0}^{m} p_k u_{n-k} = 0, \quad m \geq 2,\ p_0 \neq 0,\ p_m \neq 0, \tag{1}$$

we consider a non-homogeneous equation as follows

$$\sum_{k=0}^{m} p_k u(x-k) = v(x), \quad x \in \mathbf{R}. \tag{2}$$

For $x = n$ and $v(\mathbf{Z}) = \{0\}$ we reobtain (1).

What happens if the left side of relation (2) contains an infinity of terms? In this paper we would like to study an equation of the following form

$$\sum_{k=-\infty}^{\infty} p_k u(x-k) = v(x), \quad x \in \mathbf{R}. \tag{3}$$

This equation raises new challanges such as: in which space of functions must we search the solutions and what kind of conditions must the sequence $(p_k)_{k\in\mathbf{Z}}$ fulfil

Received by the editors: June 12, 1997.

1991 *Mathematics Subject Classification.* 42C15, 11B37.

Key words and phrases. functional equations, reccurence relations.

to be compatible. Is such a research artificial or already necessary in the mathematical landscape? In the next section we will detail upon how such an equation can appear and the importance it takes. For this, we take a trip in the world of wavelets which represents a happy marriage between the results of the signal proccesing and the results in multiresolution analysis. Further on, we will list and prove some properties both of the function and the sequence which are involved in (3). In the last section we will relate the announced study under the assumption that the sequence has a finite support.

2. A sea of wavelets without water

We try to present the notion of wavelets. The standard references for this topic are Chui [1], Daubechies [2], Meyer [4] and the literature cited here. If we denote by $L_2(\mathbf{R})$ the space of square integrable functions defined on $\mathbf{R}$, we will refer to a function $f \in L_2(\mathbf{R})$ as being a signal with finite energy given by its norm $\|f\| = (f,f)^{1/2}$ We recall that the inner product of this space is defined by $(f,g) = \int_{\mathbf{R}} f(x)\overline{g(x)}dx$. It is well-known [3] that the Fourier transform of a function $f \in L_2(\mathbf{R})$ is given by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\xi x} f(x)dx$$

and the inverse Fourier transform is

$$f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi x} \widehat{f}(\xi)d\xi.$$

For a given function f we will use throughout the paper the notation $f_{j,k}(x) := 2^{j/2} f(2^j x - k)$. For any $j, k \in \mathbf{Z}$ we can write

$$\|f(2^j \cdot -k)\| = \left\{ \int_{\mathbf{R}} |f(2^j x - k)|^2 dx \right\}^{1/2} = 2^{-j/2}\|f\|.$$

This implies $\|f_{j,k}\| = \|f\|$, $j,k \in \mathbf{Z}$.

A multiresolution analysis (MRA) of $L_2(\mathbf{R})$ is defined as a sequence of closed subspaces V_j, $j \in \mathbf{Z}$, of $L_2(\mathbf{R})$ which enjoy the following properties

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$,

(ii) $\bigcup_{j\in\mathbf{Z}} V_j$ is dense in $L_2(\mathbf{R})$ and $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$,

(iii) $v \in V_0 \;\Leftrightarrow\; v(\cdot + 1) \in V_0$,

$v \in V_j \;\Leftrightarrow\; v(2\cdot) \in V_{j+1}$, $j \in \mathbf{Z}$,

(iv) a function $\phi \in V_0$ exists such as the set $\{\phi_{0,k} : k \in \mathbf{Z}\}$ is a Riesz basis of V_0.

In accordance with this definition, if the subspace V_0 is generated by a single function $\phi \in L_2(\mathbf{R})$, that is $V_0 = \text{closure}_{L_2(\mathbf{R})}\langle \phi_{0,k} : k \in \mathbf{Z}\rangle$, then all the subspaces V_j are also generated by the same ϕ namely $V_j = \text{closure}_{L_2(\mathbf{R})}\langle \phi_{j,k} : k \in \mathbf{Z}\rangle$. In fact, the set of functions $\{\phi_{j,k} : l \in \mathbf{Z}\}$ is a Riesz basis of V_j. We will name ϕ "the scaling function" or more suggestively "the father function". It is said that ϕ generates an MRA $\{V_j\}$ of $L_2(\mathbf{R})$. Since $\phi \in V_0 \subset V_1$ and $\{\phi_{1,k} : k \in \mathbf{Z}\}$ is a Riesz basis of V_1, consequently there exists a unique l^2-sequence $(p_k)_{k\in\mathbf{Z}}$ which describes ϕ that is $\phi(x) = \sum_{k\in\mathbf{Z}} p_k \phi_{1,k}(x)$. In other words, the father function satisfies the dilation equation

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k), \quad x \in \mathbf{R}. \tag{4}$$

This also called a "two-scale relation" of the function ϕ. The sequence $(p_k)_{k\in\mathbf{Z}}$ becomes the "two-scale sequence" of ϕ. Naturally, we consider that $\sqrt{2\pi}\hat{\phi}(0) = \int_{\mathbf{R}} \phi(x)dx$ is not zero. By integrating the relation (4) over $\mathbf{R}$, we can write

$$\int_{\mathbf{R}} \phi(x)dx = \sum_{k=-\infty}^{\infty} p_k \int_{\mathbf{R}} \phi(2x-k)dx = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k \int_{\mathbf{R}} \phi(y)dy.$$

This leads us to the following identity

$$\sum_{k=-\infty}^{\infty} p_k = 2. \tag{5}$$

At this point, we introduce W_j, the orthogonal complement space of V_j in V_{j+1}, so that $V_{j+1} = V_j \oplus W_j$. We deduce that W_j, $j \in \mathbf{Z}$, are mutually orthogonal and

$$\bigoplus_{j=-\infty}^{\infty} W_j = L_2(\mathbf{R}).$$

Assuming that integer translates of ϕ generate an orthogonal basis (o.n.b.) for V_0, there exists a function $\psi \in W_0$ such as $\{\psi_{0,k} : k \in \mathbf{Z}\}$ forms an o.n.b. for W_0. At this moment the "mother wavelet" is born. Like the father ϕ generated an o.n.b. for V_j, the mother ψ generates an o.n.b. for the orthocomplements W_j of V_j, $j \in \mathbf{Z}$. It results that $\{\psi_{j,k} : (j,k) \in \mathbf{Z}\times\mathbf{Z}\}$ is an o.n.b. for $L_2(\mathbf{R})$. In fact, ψ can be constructed as follows:

$$\psi(x) = \sum_{k=-\infty}^{\infty} (-1)^k \bar{c}_{1-k}\phi(2x-k) \tag{6}$$

is the general process to build wavelets bases

3. Other features of a father function

Applying the Fourier transform to (4), the dilation equation gives

$$\widehat{\phi}(\xi) = \sum_{k=-\infty}^{\infty} p_k \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \phi(2x-k)e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} p_k 2^{-1} e^{-ik\xi/2} \int_{\mathbf{R}} \phi(y)e^{-iy\xi/2}dy.$$

If we set $H(z) := \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k z^k$, we obtain the following relation

$$\widehat{\phi}(2\xi) = H(e^{-i\xi})\widehat{\phi}(\xi). \tag{7}$$

By repeating n times the relation (7) we get

$$\widehat{\phi}(2\xi) = \prod_{k=1}^{n} H(e^{-i\xi/2^{k-1}})\widehat{\phi}(\xi/2^{n-1}).$$

Since $\widehat{\phi}$ is a continuous function on $\mathbf{R}$ and assuming that $\widehat{\phi}(0) = 1$, we easily deduce that

$$\widehat{\phi}(\xi) \to \prod_{k=1}^{\infty} H(e^{-i\xi/2^k}),$$

pointwise.

Proposition 1. *If ϕ defines an o.n.b. in V_0 the one has*

$$|h(\xi)|^2 + |h(\xi+\pi)|^2 = 1, \tag{8}$$

where $h(\xi) = H(e^{-i\xi})$.

Proof. We can write successively:

$$\delta_{0,n} = (\phi_{0,0}, \phi_{0,n}) = \int_{\mathbf{R}} \phi(x)\overline{\phi}(x-n)dx = \int_{\mathbf{R}} e^{-in\xi}|\widehat{\phi}(\xi)|^2 d\xi,$$

where we have used the Parseval identity. On the other hand,

$$\int_{\mathbf{R}} e^{-in\xi}|\widehat{\phi}(\xi)|^2 d\xi = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{2(k+1)\pi} e^{-in\xi}|\widehat{\phi}(\xi)|^2 d\xi = \int_0^{2\pi} e^{-in\xi}\left\{\sum_{k=-\infty}^{\infty} |\widehat{\phi}(\xi+2k\pi)|^2\right\}d\xi$$

Because $\frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi}d\xi = \delta_{0,n}$, the above relations lead us to the following identity

$$\sum_{k=-\infty}^{\infty} |\widehat{\phi}(\xi+2k\pi)|^2 = \frac{1}{2\pi}.$$

We choose $\xi := 2\xi$ and according to (7), we have

$$\frac{1}{2\pi} = \sum_{k=-\infty}^{\infty} |\widehat{\phi}(2\xi+2k\pi)|^2 = \sum_{k=-\infty}^{\infty} |H(e^{-i(\xi+k\pi)})|^2|\widehat{\phi}(\xi+k\pi)|^2 = \sum_{k\ \text{even}} + \sum_{k\ \text{odd}} =$$

$$= |H(e^{-i\xi})|^2 \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2 + |H(e^{-i(\xi+\pi)}| \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + (2k+1)\pi)|^2 =$$

$$= \frac{1}{2\pi}(|h(\xi)|^2 + |h(\xi+\pi)|^2).$$

We have used the fact that h is 2π-periodic.

Taking into account (5) we get $h(0) = H(1) = 1$. By using (8), it results $h(\pi) = H(-1) = 0$, in other words $\sum_{k=-\infty}^{\infty} (-1)^k p_k = 0$. We are now able to state another property in connection with equation (4).

Proposition 2. *If ϕ defines an o.n.b. in V_0 then the following identities*

$$\sum_{k\in\mathbf{Z}} p_{2k} = \sum_{k\in\mathbf{Z}} p_{2k+1} = 1$$

hold.

Proposition 3. *If ϕ is normalized, that is $\int_{\mathbf{R}} \phi(x)dx = 1$, then the following identities*

$$\sum_{k\in\mathbf{Z}} \phi(x-k) = \sum_{k\in\mathbf{Z}} \phi(k) = 1$$

hold.

Proof. If we put $s(x) = \sum_{k\in\mathbf{Z}} \phi(x-k)$, by using the dilation equation, we can write

$$s(x) = \sum_{k=-\infty}^{\infty} \left\{ \sum_{n\in\mathbf{Z}} p_n \phi(2x - 2k - n) \right\} = \sum_{k=-\infty}^{\infty} \left\{ \sum_{n \text{ even}} + \sum_{n \text{ odd}} \right\} =$$

$$= \sum_{k=-\infty}^{\infty} \left\{ \sum_{m\in\mathbf{Z}} p_{2m}\phi(2x - 2(k+m)) + \sum_{m\in\mathbf{Z}} p_{2m+1}\phi(2x - 2(k+m) - 1) \right\} =$$

$$= \sum_{l=-\infty}^{\infty} \phi(2x-2l) \left(\sum_{m\in\mathbf{Z}} p_{2m} \right) + \sum_{l=-\infty}^{\infty} \phi(2x-2l-1) \left(\sum_{m\in\mathbf{Z}} p_{2m+1} \right) = \sum_{l=-\infty}^{\infty} \phi(2x-l) = s(2x).$$

We have used Proposition 2. In this way, we have obtained $s(x) = s(2x)$ which implies $\widehat{s}(\xi) = 2\widehat{s}(2\xi)$. This represents a dilation equation with $p_0 = 2$ and all other coefficients are zero. The non trivial solution is $s = \delta$, the Dirac delta function ([3], Leçon n° 31). We deduce that s is a constant. Taking $\sum_{k=-\infty}^{\infty} \phi(x-k) = \alpha$ and integrating over [0,1] we have

$$\alpha = \sum_{k=-\infty}^{\infty} \int_0^1 \phi(x-k)dx = \sum_{k=-\infty}^{\infty} \int_{-k}^{1-k} \phi(y)dy = \int_{\mathbf{R}} \phi(y)dy = 1.$$

For $x = 0$, it holds $1 = \sum_{k=-\infty}^{\infty} \phi(-k) = \sum_{k\in\mathbf{Z}} \phi(k)$ which completes the proof.

4. Particular approach

We are going to study a two-scale relation which is described by finite sums. We suppose that the integers $N' < N''$ exist, such as

$$(i)\quad p_{N'} \neq 0,\ p_{N''} \neq 0 \qquad (ii)\quad p_k = 0 \text{ for } k < N' \text{ and } k > N''. \tag{9}$$

We will only be concerned with scaling functions which are continuous everywhere. Because $\phi \in L_1(\mathbf{R}) \cap C(\mathbf{R})$ we are looking for the solutions ϕ with bounded support. Firstly, we specify that a general method for constructing the scale function ϕ is by using iterates and which does not involve $\widehat{\phi}$. In fact, ϕ solves (4) if $T(\phi) = \phi$ where $T(\phi) = \sum_{k\in\mathbf{Z}} p_k\phi(2x-k)$. We try to find this fixed point as usual: find a suitable ϕ_0, define $\phi_n = T^n\phi_0$, and prove that ϕ_n has a limit. In this way, $\phi(x) = \lim_{n\to\infty}\phi_n(x)$. As a consequence of this recursive scheme we can expose

Proposition 4. *If the relation (9) is fulfilled then*

$$\operatorname{supp}\phi \subset [N', N''] \quad \textit{and} \quad \operatorname{supp}\psi \subset \left[\frac{1+N'-N''}{2}, \frac{1-N'+N''}{2}\right]$$

hold, where ϕ and ψ satisfy the equations (4) respectively (6).

Proof. We use the recursive scheme and choose ϕ_0 with compact support. Let's take $\operatorname{supp}\phi_0 = [N_0', N_0'']$. Successive applications of T define

$$\phi_{j+1}(x) = (T\phi_j)(x) = \sum_{k=N'}^{N''} p_k\phi_j(2x-k). \tag{10}$$

We have $\operatorname{supp}\phi_1 = \left[\frac{N_0'+N'}{2}, \frac{N_0''+N''}{2}\right]$ and denoting $\operatorname{supp}\phi_j = [N_j', N_j'']$ it results $N_{j+1}' = (N_j'+N')/2$, $N_{j+1}'' = (N_j''+N'')/2$. By computations, it follows

$$N_j' = \frac{N_0'}{2^j} + \left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^j}\right)N', \quad N_j'' = \frac{N_0''}{2^j} + \left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^j}\right)N'',$$

and consequently $\lim_{j\to\infty} N_j' = N'$, $\lim_{j\to\infty} N_j'' = N''$. This proves that $\operatorname{supp}\phi \subset [N', N'']$. In order to obtain the second inclusion, in (9) we notice that p_{1-k} is only nonzero for $k \in [1-N'', 1-N']\cap\mathbf{Z}$. On the other hand, we have $\operatorname{supp}\phi(2\cdot -k) \subset \left[\frac{N'+k}{2}, \frac{N''+k}{2}\right]$. relation (6) allows us to obtain the desired result.

Investigating (9) we remark that by a change of index in p_k, the relation (4) be written as follows

$$\phi(x) = \sum_{k=0}^{N} p_k\phi(2x-k), \quad p_0p_N \neq 0. \tag{11}$$

Of course, the scaling function ϕ must also be shifted accordingly.

Of the previous theorem, $\text{supp}\phi \subset [0, N]$ and knowing that $\phi \in C(\mathbf{R})$ we deduce $\phi(0) = \phi(N) = 0$. Firstly, we need to determine $\phi(k)$, $1 \le k \le N-1$. Substituting $x = k$, $1 \le k \le N-1$, into (11) leads to $N-1$ linear equations with the $N-1$ unknowns $\phi(k)$. In matrix notation we have

$$v = Pv, \tag{12}$$

where v is the column vector $(\phi(1), \phi(2), \ldots, \phi(N-1))^T$ and P the $(N-1) \times (N-1)$ matrix

$$P := (p_{2j-k})_{1 \le j,k \le N-1} \tag{13}$$

with j being the row index and k the column index. Recalling that ϕ generates a partition of unity (see Proposition 3) we can determine the values of $\phi(k)$, $k \in \mathbf{Z}$, by finding the eigenvector v corresponding to the eigenvalue 1 and imposing $\sum_{k=1}^{N-1} \phi(k) = 1$. Define ϕ_0 to be the piecewise linear function which takes exactly the values $\phi(k)$ at the integers, that is

$$\phi_0(x) := \phi(x)(k+1-x) + \phi(k+1)(x-k), \quad x \in [k, k+1].$$

We compute ϕ_j, by using (10) and it follows that ϕ_j are piecewise linear with nodes at $k/2^j \in [0, N]$, $k \in \mathbf{Z}$.

Let's make some examples. For $N = 3$, it is known the quadratic cardinal B-spline N_3 whose two-scale equation is

$$N_3(x) = \frac{1}{4}N_3(2x) + \frac{3}{4}N_3(2x-1) + \frac{3}{4}N_3(2x-2) + \frac{1}{4}N_3(2x-3).$$

However, there is another alternative [2], namely Daubechies'scaling function ϕ^D governed by

$$\phi^D(x) = \frac{1+\sqrt{3}}{4}\phi^D(2x) + \frac{3+\sqrt{3}}{4}\phi^D(2x-1) + \frac{3-\sqrt{3}}{4}\phi^D(2x-2) + \frac{1-\sqrt{3}}{4}\phi^D(2x-3).$$

In the following, we choose $p_0 = \mu$ and $p_1 = -\frac{1}{\mu}$ where $\mu = \left(1+\sqrt{5}\right)/2$. In concordance with Proposition 2, we must take $p_3 = \mu$ and $p_2 = 1 - \mu$. Thus the matrix P defined in (13) becomes

$$P = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix}.$$

The solution of (12) is $v = a \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and the normalization condition implies $a = 1/2$. One obtains $\phi(1) = \phi(2) = 1/2$.

Using the golden ratio in the componence of the matrix P we get a nice scale function which is defined on $\mathbf{Z}$ as follows

$$\phi(k) = \begin{cases} 1/2, & k \in \{1,2\} \\ 0, & k \in \mathbf{Z} \setminus \{1,2\}. \end{cases}$$

Having the values of $\phi(k)$ it is now easy to compute $\phi(k/2^j)$, $(k,j) \in \mathbf{Z} \times \mathbf{Z}$. In fact, the Interpolatory Graphical Display Algorithm can be applied, see [1], page 93.

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