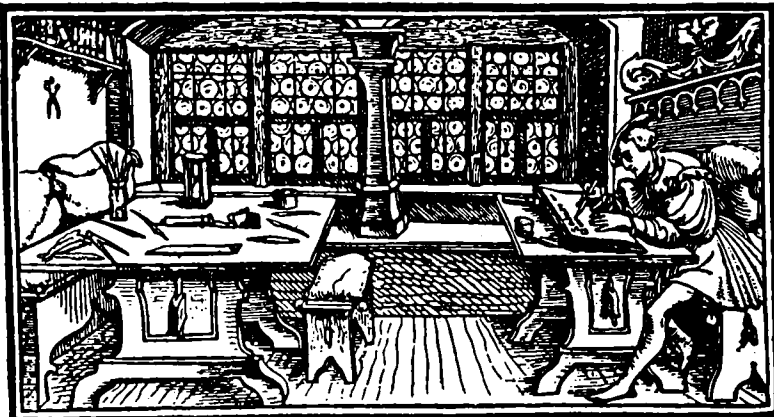


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A BIVARIATE EXTENSION OF THE BERNSTEIN POLYNOMIALS

OCTAVIAN AGRATINI

Abstract. In this paper it is given an extension to two variables of the Bernstein $B_{n,r}f$ operators and there are investigated their approximation properties.

1. In the paper [2] is introduced and studied a new sequence of Bernstein type polynomials. Namely, for any function $f \in C^r[0, 1]$ is defined the associated polynomial:

$$(B_{n,r}f)(x) = \sum_{k=0}^n \sum_{i=0}^r \frac{f^{(i)}\left(\frac{k}{n}\right)}{i!} \left(x - \frac{k}{n}\right)^i \binom{n}{k} x^k (1-x)^{n-k}. \quad (1)$$

It has been proved the estimation:

$$\|f - (B_{n,r}f)\| = O\left(n^{-\frac{r}{2}} \omega\left(f^{(r)}; n^{-1/2}\right)\right),$$

where $\|g\| = \sup\{|g(x)| : x \in [0, 1]\}$ for $g \in C[0, 1]$ and ω is the modulus of continuity of the function f , defined as usually:

$$\omega(f; s) = \sup\{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq s\}.$$

The aim of this paper is to give an extension to two variables of the $B_{n,r}f$ and to investigate their approximation properties. It should be mentioned that there are many extensions to two variables of the linear operators of approximation (see, e.g. [3], [4], [5]).

2. Let $E = [0, 1] \times [0, 1]$ and $f : E \rightarrow \mathbf{R}$ differentiable of order r on E . The Taylor polynomial of degree r associated to the function f , in a point $(a, b) \in E$, is defined by:

$$(T_r f)(a, b; x, y) = f(a, b) + \frac{1}{1!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right) f(a, b) + \\ + \frac{1}{2!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right)^2 f(a, b) + \dots + \frac{1}{r!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right)^r f(a, b).$$

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The corresponding Taylor approximation formula is

$$f(x, y) = (T_r f)(a, b; x, y) + (R_r f)(a, b; x, y). \quad (2)$$

We introduce the notations:

$$(D^k f)(a, b; x, y) = \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right)^k f(a, b) = \sum_{i=0}^k \binom{k}{i} \frac{\partial^k f(a, b)}{\partial x^{k-i} \partial y^i} (x-a)^{k-i} (y-b)^i.$$

Then the formula (2) becomes:

$$f(x, y) = f(a, b) + \sum_{i=1}^r \frac{1}{i!} (D^i f)(a, b; x, y) + (R_r f)(a, b; x, y).$$

Definition. A generalized Bernstein polynomial of two variables of order $(n, m; r)$ for a differentiable function f of order r on E is a polynomial having the form:

$$(B_{n,m}^{(r)} f)(x, y) = \sum_{k=0}^n \sum_{l=0}^m (T_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) \binom{n}{k} \binom{m}{l} (1-x)^{n-k} x^k (1-y)^{m-l} y^l,$$

where

$$(T_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) = \sum_{i=0}^r \frac{1}{i!} (D^i f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right).$$

It is evident that for $r = 0$

$$(T_0 f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) = f \left(\frac{k}{n}, \frac{l}{m} \right)$$

and consequently $B_{n,m}^{(0)} f$ becomes the classical polynomial of Bernstein of two variables.

3. We intend to evaluate the quantity:

$$(E_{n,m}^{(r)} f)(x, y) = |f(x, y) - (B_{n,m}^{(r)} f)(x, y)|.$$

We need to use the identity:

$$\sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} x^k (1-x)^{n-k} y^l (1-y)^{m-l} = 1.$$

By multiplying it by $f(x, y)$ we can write successively:

$$\begin{aligned} (E_{n,m}^{(r)} f)(x, y) &\leq \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \left| f(x, y) - (T_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) \right| x^k (1-x)^{n-k} y^l (1-y)^{m-l} = \\ &= \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \left| (R_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) \right| x^k (1-x)^{n-k} y^l (1-y)^{m-l}. \end{aligned} \quad (3)$$

It is known that if f has continuous partial derivatives of order r in a neighbourhood of $(\frac{k}{n}, \frac{l}{m})$, then the remainder can be expressed under the form:

$$(R_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) = \frac{1}{r!} s \left(\frac{k}{n}, \frac{l}{m} \right) (x, y) \rho_{\left(\frac{k}{n}, \frac{l}{m} \right)}^r (x, y), \quad (4)$$

where $s \left(\frac{k}{n}, \frac{l}{m} \right)$ is a continuous in $(\frac{k}{n}, \frac{l}{m})$ and vanishes in this point. Also, we notice that:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m} \right)} (x, y) = \left(\left(x - \frac{k}{n} \right)^2 + \left(y - \frac{l}{m} \right)^2 \right)^{1/2}$$

In the next stage we shall consider that the remainder of Taylor's formula expressed by (4) fulfills the following condition: exist the real constants A, B and the numbers $p \geq 2, q \geq 2$ so as

$$|s \left(\frac{k}{n}, \frac{l}{m} \right) (x, y)| \leq A \left| x - \frac{k}{n} \right|^p + B \left| y - \frac{l}{m} \right|^q, \quad (x, y) \in E. \quad (5)$$

Because (x, y) and $(\frac{k}{n}, \frac{l}{m})$ belong to E , we can deduce:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m} \right)} (x, y) \leq 2^{\frac{5}{2}}. \quad (6)$$

From (4), (5) and (6) we get:

$$\left| (R_r f) \left(\frac{k}{n}, \frac{l}{m}; x, y \right) \right| \leq \frac{2^{\frac{5}{2}}}{r!} \left(A \left| x - \frac{k}{n} \right|^p + B \left| y - \frac{l}{m} \right|^q \right) \leq \frac{2^{\frac{5}{2}}}{r!} \left(A \left(x - \frac{k}{n} \right)^2 + \left(y - \frac{l}{m} \right)^2 \right).$$

The following inequalities are well-known:

$$\sum_{k=0}^n \sum_{l=0}^m (k - nx)^2 \binom{n}{k} \binom{m}{l} x^k (1-x)^{n-k} y^l (1-y)^{m-l} \leq \frac{n}{4}$$

and

$$\sum_{k=0}^n \sum_{l=0}^m (l - my)^2 \binom{n}{k} \binom{m}{l} x^k (1-x)^{n-k} y^l (1-y)^{m-l} \leq \frac{n}{4}.$$

They lead to the next result:

Theorem. *Let $f : E \rightarrow \mathbf{R}$ with all partial derivatives of order r continuous on E . If the remainder of Taylor's formula fulfills the condition (5) then we obtain the inequality:*

$$|f(x, y) - (B_{m,n}^{(r)} f)(x, y)| \leq \frac{2^{\frac{5}{2}}}{4r!} \left(\frac{A}{n} + \frac{B}{m} \right).$$

Corollary. *Under the hypothesis of this theorem we can further write:*

$$\lim_{m,n \rightarrow \infty} \|f - B_{m,n}^{(r)} f\| = 0,$$

where $\|\cdot\| = \max_E |\cdot|$.

4. We mention that in [1] we have introduced a class of linear polynomial approximating operators $(L_{nr})_{n \geq 1}$, $r = 0, 1, 2, \dots$ for the functions $f \in C^r[0, 1]$. In order to construct them we used the Taylor polynomial of degree r and a class of linear positive operators generated by a probabilistic method. Also, we studied the order of approximation using the moduli of continuity of first and second order, $(L_{nr}f)_{n \geq 1}$ including as a special case the generalized Bernstein polynomials defined in [2].

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