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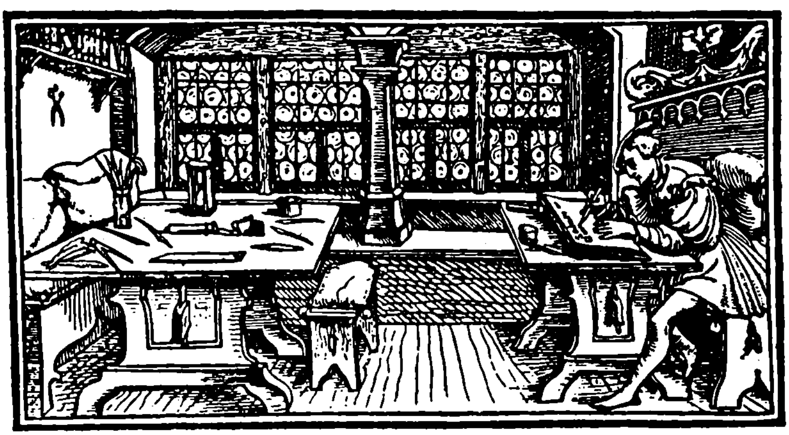

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A BIVARIATE EXTENSION OF THE BERNSTEIN POLYNOMIALS

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Abstract. In this paper it is given an extension to two variables of the Bernstein $B_{n,r}f$ operators and there are investigated their approximation properties.

1. In the paper [2] is introduced and studied a new sequence of Bernstein type polynomials. Namely, for any function $f \in C^r[0,1]$ is defined the associated polynomial:

$$(B_{n,r}f)(x) = \sum_{k=0}^{n}\sum_{i=0}^{r} \frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x - \frac{k}{n}\right)^i \binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

It has been proved the estimation:

$$\|f - (B_{n,r}f)\| = O\left(n^{-\frac{r}{2}}\omega\left(f^{(r)}; n^{-1/2}\right)\right),$$

where $\|g\| = \sup\{|g(x)| : x \in [0,1]\}$ for $g \in C[0,1]$ and ω is the modulus of continuity of the function f, defined as usually:

$$\omega(f;s) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x-y| \le s\}.$$

The aim of this paper is to give an extension to two variables of the $B_{n,r}f$ and to investigate their approximation properties. It should be mentioned that there are many extensions to two variables of the linear operators of approximation (see, e.g. [3], [4], [5]).

2. Let $E = [0,1] \times [0,1]$ and $f : E \to \mathbf{R}$ differentiable of order r on E. The Taylor polynomial of degree r associated to the function f, in a point $(a,b) \in E$, is defined by:

$$(T_r f)(a,b;x,y) = f(a,b) + \frac{1}{1!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right) f(a,b) +$$

$$+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{r!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^r f(a,b).$$

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The corresponding Taylor approximation formula is

$$f(x,y) = (T_r f)(a,b;x,y) + (R_r f)(a,b;x,y). \tag{2}$$

We introduce the notations:

$$(D^k f)(a,b;x,y) = \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^k f(a,b) = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f(a,b)}{\partial x^{k-i}\partial y^i}(x-a)^{k-i}(y-b)^i.$$

Then the formula (2) becomes:

$$f(x,y) = f(a,b) + \sum_{i=1}^{r} \frac{1}{1!}(D^i f)(a,b;x,y) + (R_r f)(a,b;x,y).$$

Definition. A generalized Bernstein polynomial of two variables of order $(n,m;r)$ for a differentiable function f of order r on E is a polynomial having the form:

$$(B_{n,m}^{(r)} f)(x,y) = \sum_{k=0}^{n}\sum_{l=0}^{m} (T_r f)\left(\frac{k}{n},\frac{l}{m};x,y\right)\binom{n}{k}\binom{m}{l}(1-x)^{n-k}x^k(1-y)^{m-l}y^l,$$

where

$$(T_r f)\left(\frac{k}{n},\frac{l}{m};x,y\right) = \sum_{i=0}^{r}\frac{1}{i!}(D^i f)\left(\frac{k}{n},\frac{l}{m};x,y\right).$$

It is evident that for $r=0$

$$(T_0 f)\left(\frac{k}{n},\frac{l}{m};x,y\right) = f\left(\frac{k}{n},\frac{l}{m}\right)$$

and consequently $B_{n,m}^{(0,)} f$ becomes the classical polynomial of Bernstein of two variables.

3. We intend to evaluate the quantity:

$$(E_{n,m}^{(r)} f)(x,y) = |f(x,y) - (B_{n,m}^{(r)} f)(x,y)|.$$

We need to use the identity:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l} = 1.$$

By multiplying it by $f(x,y)$ we can write successively:

$$(E_{n,m}^{(r)} f)(x,y) \le \sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\left|f(x,y) - (T_r f)\left(\frac{k}{n},\frac{l}{m};x,y\right)\right| x^k(1-x)^{n-k}y^l(1-y)^{m-l} =$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}\left|(R_r f)\left(\frac{k}{n},\frac{l}{m};x,y\right)\right| x^k(1-x)^{n-k}y^l(1-y)^{m-l}. \tag{3}$$

It is known that if f has continuous partial derivatives of order r in a neighbourhood of $(\frac{k}{n}, \frac{l}{m})$, then the remainder can be expressed under the form:

$$(R_r f)\left(\frac{k}{n}, \frac{l}{m}; x, y\right) = \frac{1}{r!} s_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) \rho^r_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y), \tag{4}$$

where $s_{\left(\frac{k}{n}, \frac{l}{m}\right)}$ is a continuous in $(\frac{k}{n}, \frac{l}{m})$ and vanishes in this point. Also, we notice that:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) = \left(\left(x - \frac{k}{n}\right)^2 + \left(y - \frac{l}{m}\right)^2\right)^{1/2}$$

In the next stage we shall consider that the remainder of Taylor's formula expressed by (4) fulfills the following condition: exist the real constants A, B and the numbers $p \geq 2$, $q \geq 2$ so as

$$|s_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y)| \leq A\left|x - \frac{k}{n}\right|^p + B\left|y - \frac{l}{m}\right|^q, \quad (x, y) \in E. \tag{5}$$

Because (x, y) and $(\frac{k}{n}, \frac{l}{m})$ belong to E, we can deduce:

$$\rho_{\left(\frac{k}{n}, \frac{l}{m}\right)}(x, y) \leq 2^{\frac{r}{2}}. \tag{6}$$

From (4), (5) and (6) we get:

$$\left|(R_r f)\left(\frac{k}{n}, \frac{l}{m}; x, y\right)\right| \leq \frac{2^{\frac{r}{2}}}{r!}\left(A\left|x - \frac{k}{n}\right|^p + B\left|y - \frac{l}{m}\right|^q\right) \leq \frac{2^{\frac{r}{2}}}{r!}\left(A\left(x - \frac{k}{n}\right)^2 + \left(y - \frac{l}{m}\right)^2\right).$$

The following inequalities are well-known:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(k - nx)^2\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l} \leq \frac{n}{4}$$

and

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(l - my)^2\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}y^l(1-y)^{m-l} \leq \frac{n}{4}.$$

They lead to the next result:

Theorem. *Let $f : E \to \mathbf{R}$ with all partial derivatives of order r continuous on E. If the remainder of Taylor's formula fulfills the condition (5) then we obtain the inequality:*

$$|f(x, y) - (B^{(r)}_{m,n} f)(x, y)| \leq \frac{2^{\frac{r}{2}}}{4r!}\left(\frac{A}{n} + \frac{B}{m}\right).$$

Corollary. *Under the hypothesis of this theorem we can further write:*

$$\lim_{m,n \to \infty} ||f - B^{(r)}_{m,n} f|| = 0,$$

where $\|\cdot\| = \max_E |\cdot|$.

4. We mention that in [1] we have introduced a class of linear polynomial approximating operators $(L_{nr})_{n\geq 1}$, $r = 0, 1, 2, \ldots$ for the functions $f \in C^r[0,1]$. In order to construct them we used the Taylor polynomial of degree r and a class of linear positive operators generated by a probabillistic method. Also, we studied the order of approximation using the moduli of continuity of first and second order, $(L_{nr}f)_{n\geq 1}$ including as a special case the generalized Bernstein polynomials defined in [2].

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