ON MODIFIED BETA OPERATORS

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Dedicated to Professor D.D. Stancu on his 70^{th} birthday

Abstract. In the present paper, we deal with an integral operator of beta-type $L_{m,y}$ depending on a positive real parameter y. We give an estimation of the order of approximation by using the first order modulus of continuity. Also, we prove an asymptotic formula of Voronovskaja type and we show that the operator preserves the Lipschitz constants.

1. Introduction

In time, several integral operators which are associated with beta-type probability distributions have been discussed and interesting properties have been proved. In this respect, we mention the papers [1], [2], [5], [6], [7].

Let us denote by $L_B[0,\infty)$ the linear space of real bounded functions defined on $[0,\infty)$ and Lebesgue measurable. In [8] D.D. Stancu introduced a new beta second-kind approximating operator defined on $L_B[0,\infty)$ as:

$$(L_m f)(x) = L_m(f(t),x) = \frac{1}{B(mx,m+1)}\int_0^\infty f(t)\frac{t^{mx-1}}{(1+t)^{mx+m+1}}dt, \quad x>0, \qquad (1)$$

and $(L_m f)(0) = f(0)$, $B(\cdot,\cdot)$ being the beta function.

This is an integral linear positive operator of Feller type which reproduces the linear

functions. Starting from L_m defined by (1), we introduce and investigate a sequence of linear positive operators depending on a parameter $y>0$. These modified beta operators are defined as follows:

$$(L_{m,y} f)(x) = \frac{1}{B(my,m+1)}\int_0^\infty f(t+x)\frac{t^{my-1}}{(1+t)^{my+m+1}}dt, \quad x\ge 0, \qquad (2)$$

where $f \in L_B[0,\infty)$.

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It is clear that for any $x > 0$ we have:

$$(L_{m,x}f)(0) = (L_m f)(x).$$

This type of construction was used in the papers [3] and [4] for an integra modification of Szász operators.

The next section provides the main results of this paper. All the proofs and th necessary supporting results are provided in section three.

2. Main results

Theorem 1. *Let $a > 0$ and $f \in C[0,\infty)$ such that $L_{m,y}(|f|, x) < \infty$. For any $m > 1$ and $y \in [0,a]$ we have the following inequality:*

$$|(L_{m,y}f)(x) - f(x+y)| \le \left(1 + \sqrt{a(a+1)}\right)\omega\left(f, \frac{1}{\sqrt{m-1}}\right), \tag{3}$$

where $\omega(f,\cdot)$ represents the modulus of continuity of the function f.

Corollary 1. *Let $0 < a < \infty$. If $f \in L_B[0,\infty) \cap C[0,2a]$, then, for any fixed $y \in [0,a]$, we have:*

$$\lim_{m\to\infty} (L_{m,y}f)(x) = f(x+y), \tag{4}$$

uniformly in $[0,a]$.

Theorem 2. *Let $a > 0$ and $f \in L_B[0,\infty)$ differentiable in some neighborhood of a point $x + y \in [0,a]$ such that at this point f'' exists. Then we have:*

$$\lim_{m\to\infty} m(f(x+y) - (L_{m,y}f)(x)) = -\frac{y(y+1)}{2} f''(x+y). \tag{5}$$

Theorem 3. *Let $a > 0$ and $f \in L_B[0,\infty)$. Then $f \in \mathrm{Lip}_{[0,a]}(A,\mu)$ if and only if $L_{m,y}f \in \mathrm{Lip}_{[0,a]}(A,\mu)$, where $A > 0$ and $\mu \in (0,1]$.*

It is appropriate to remark here that $g \in \mathrm{Lip}_{[0,a]}(A,\mu)$ if for any x and y belonging to $[0,a]$ we have:

$$|g(x) - g(y)| \le A|x-y|^{\mu}. \tag{6}$$

3. Proofs

Before we proceed with the proofs, we recall some useful relations:

$$(L_{m,y}e_0)(x) = (L_m e_0)(y) = 1, \tag{7}$$

$$(L_{m,y}e_1)(x) = x + y, \tag{8}$$

$$L_m((t-x)^2; x) = \frac{x(x+1)}{m-1}, \tag{9}$$

where $e_k(t) = t,\ t \geq 0,\ k = 0, 1.$

Proof of Theorem 1. We have

$$|(L_{m,y}f)(x) - f(x+y)| \leq$$
$$\leq \frac{1}{B(my, m+1)} \int_0^\infty |f(t+x) - f(y+x)| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt.$$

It is verified that for any $\delta > 0$

$$|f(t+x) - f(y+x)| \leq \left(1 + \frac{1}{\delta}|t-y|\right) \omega(f, \delta).$$

From the above relations, by making use of the Cauchy inequality and of the relations (7), (8), (9) we can write successively:

$$|(L_{m,y}f)(x) - f(x+y)| \leq$$
$$\leq \left(1 + \frac{1}{\delta} \frac{1}{B(my, m+1)} \int_0^\infty |t-y| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt\right) \omega(f, \delta) \leq$$
$$\leq \left(1 + \frac{1}{\delta} L_m^{1/2}((t-y)^2; y)\right) \omega(f, \delta) = \left(1 + \frac{1}{\delta}\sqrt{\frac{y(y+1)}{m-1}}\right) \omega(f, \delta).$$

But $y \leq a$ and if we take $\delta = 1/\sqrt{m-1}$, we arrive at inequality (3). □

Proof of Theorem 2. Because f has a finite second order derivative at a point $x+y \in [0, a]$ then f can be expanded by Taylor's formula:

$$f(t) = f(x+y) + (t-x-y)f'(x+y) + \frac{(t-x-y)^2}{2} f''(x+y) + (t-x-y)^2 r_{2,y}(t),$$

where $r_{2,y}$ is a real valued function having the property: $r_{2,y}(t) \to 0$ as $t \to x+y$. Using (7) and (8), we get:

$$(L_{m,y}f)(x) - f(x+y) = \frac{y(y+1)}{2(m-1)} f''(x+y) + R_{2,y}(x),$$

where $R_{2,y}$ is given by:

$$R_{2,y}(x) = \frac{1}{B(my, m+1)} \int_0^\infty (t-y)^2 r_{2,y}(t+x) \frac{t^{my-1}}{(1+t)^{my+m+1}} dt.$$

Taking into account that $r_{2,y}(t+x)$ tends to zero when t tends to y, it follows that for every $\varepsilon > 0$ there exists an $\delta > 0$ so that for every t for which $|t-y| < \delta$, we have $|r_{2,y}(t+x)| < \varepsilon$. Since $r_{2,y}$ is bounded on $[0,a]$, for every t for which $|t-y| \geq \delta$, we deduce:

$$|r_{2,y}(t+x)| \leq M \leq M\delta^{-2}(t-y)^2.$$

Consequently, the inequality

$$|r_{2,y}(t+x)| \leq \varepsilon + M\delta^{-2}(t-y)^2$$

holds. By choosing $\varepsilon = \frac{1}{m^2}$, after few calculations, we obtain

$$\lim_{m\to\infty} mR_{2,y}(x) = 0$$

which leads us to the desired result. Further, the convergence from (5) is uniform if f'' is continuous on $[0,a]$. □

Proof of Theorem 3. Let $f \in L_B[0,\infty) \cap \mathrm{Lip}_{[0,a]}(A,\mu)$ and $x_1, x_2 \in [0,a]$ such that $x_1 + y \leq a$, $x_2 + y \leq a$. Considering (2), (6) and (7) it results

$$|(L_{m,y}f)(x_1) - (L_{m,y}f)(x_2)| \leq$$
$$\leq \frac{1}{B(my, m+1)} \int_0^\infty |f(t+x_1) - f(t+x_2)| \frac{t^{my-1}}{(1+t)^{my+m+1}} dt \leq A|x_1 - x_2|^\nu.$$

Thus, $L_{m,y}f$ preserves Lipschitz constants.

Now, we assume $L_{m,y}f \in \mathrm{Lip}_{[0,a]}(A,\mu)$. For any integer $m > 1$ and $x_i + y \in [0,a]$, $(i = 0,1)$, we can write:

$$|f(x_1+y) - f(x_2+y)| - |(L_{m,y}f)(x_1) - f(x_1+y)| - |f(x_2+y) -$$
$$- (L_{m,y}f)(x_2)| \leq |(L_{m,y}f)(x_1) - (L_{m,y}f)(x_2)| \leq A|x_1 - x_2|^\nu.$$

With the help of relation (4), we obtain easily that $f \in \mathrm{Lip}_{[0,a]}(A,\mu)$. This completes the proof of Theorem 3. □

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