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CONTINUATION PRINCIPLES FOR COINCIDENCES

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1. Introduction

This paper is devoted to the solvability of semi-linear operator equations of the form $Lx=N(x)$ in Banach spaces. We use the basic idea of the continuation methods: embed the equation in a one-parameter family of equations $Lx=H(x,\lambda)$, $\lambda\in[0,1]$, where $H(.,1)=N$ and $H(.,0)$ is suitably simple, and try to deduce the solvability of the equation for $\lambda=1$ from that corresponding to $\lambda=0$. There are two main approaches to this problem. One is based on topological invariants like the coincidence degree, see [1], [3] and [6], while the other uses the notion of an essential map. Our discussion follows the second approach.

The literature on this subject is now quite extensive, see [4], [5], [7] and [9]. In the present paper we refine and complement the existing results and we also obtain "no degree" versions of some continuation theorems by Capietto-Mawhin-Zanolin [1].

2. Continuation principles for families of operators having the same domain

Let X and Y be Banach spaces and $L:D(L)\subset X\to Y$ a linear Fredholm map of index zero, that is

$$\operatorname{Im} L \text{ is closed and } \dim\ker L=\operatorname{codim}\operatorname{Im} L<\infty.$$

Let $X=X_1\oplus X_2$ and $Y=Y_1\oplus Y_2$, where $X_1=\ker L$ and $Y_2=\operatorname{Im} L$. Let $P:X\to X_1$ and $Q:Y\to Y_1$ be continuous linear projectors and $J:X_1\to Y_1$ a fixed linear isomorphism. Then $L+JP$ is a bijective linear map.

If Z is a metric space and $N:Z\to Y$, we say that N is *L-compact* (on Z) provided that $(L+JP)^{-1}N$ is compact, i.e. it is continuous from Z into X and maps Z into a relatively compact subset of X. We say that N is *L-completely continuous* (on Z) if N is L-compact on each bounded subset of Z.

Remarks **1.** The definition of an L-compact or L-completely continuous map does not depend on the choice of J (see [3]).

2. The map $JP:X\to Y$ is L-completely continuous.

3. If $N:Z\subset X\to Y$ is L-completely continuous, then $N+JP$ is L-completely continuous too.

4. Each L-compact map is L-completely continuous. The converse is true in case that Z is bounded.

5. If $D(L)$ is closed (when we may suppose that $D(L)=X$ without loss of generality) and L is bounded, then $(L+JP)^{-1}$ is a bounded linear map and consequently, a map is L-compact or L-completely continuous if and only if it is compact or completely continuous, respectively (this case was considered in [4]).

Finally, we take a subset K_0 of X, a nonempty open bounded subset U of K_0 and a nonempty convex set $K \subset Y$. We denote by $\overline{U}$ and ∂U the closure and the boundary of U with respect to K_0. Let

$$M = \{F : \overline{U} \to K;\ F \text{ is } L-\text{compact and } Lx \neq F(x) \text{ on } \partial U\}.$$

We say that a map $F \in M$ *is essential in* M provided that for each $G \in M$ with $G|_{\partial U} = F|_{\partial U}$, there exists at least one $x \in U$ such that $Lx = G(x)$.

THEOREM 1 ([5]). *Assume* $H : \overline{U} \times [0,1] \to K$ *is L-compact on* $\overline{U} \times [0,1]$ *and denote* $H_\lambda = H(.,\lambda)$. *Also suppose*

(a) $Lx \neq H(x,\lambda)$ *for all* $x \in \partial U$ *and* $\lambda \in [0,1]$;

(b) H_0 *is essential in M.*

Then there exists $x \in U$ *such that* $Lx = H(x,1)$. *Moreover,* H_1 *is essential in M too.*

Remark **6.** For $X=Y$, $K_0=K$ and $L=I$ (the identity of X), Theorem 1 is the classical topological transversality theorem.

For an example of essential map, we have the following proposition.

PROPOSITION 1. *Suppose that* K_0 *is convex and*

$$(L+JP)^{-1}(K+JP(\overline{U})) \subset K_0. \tag{1}$$

Let $F_0 : \overline{U} \to Y_1$ *be L-compact on* $\overline{U}$ *and* $x_0 \in U$. *Assume*

$$Lx_0 + F_0(\overline{U}) \subset K \tag{2}$$

$$F_0(x) \neq 0 \ \text{ for any } x \in (x_0 + X_1) \cap \partial U \tag{3}$$

$$\langle F_0(x), J(x-x_0)\rangle \leq 0 \ \text{ for any } x \in (x_0 + X_1) \cap \partial U \tag{4}$$

where $\langle .,. \rangle$ *denotes the euclidean scalar product on* Y_1. *Then the map* $Lx_0 + F_0$ *is essential in M.*

Proof. Let $G \in M$ and $G|_{\partial U} = Lx_0 + F_0|_{\partial U}$. We have to prove the solvability in U of equation $Lx = G(x)$, or equivalently, $x = (L+JP)^{-1}(G+JP)(x)$. For this, we define $H : \overline{U} \times [0,1] \to K_0$,

$$H(x,\lambda) = (1-\lambda)x_0 + \lambda(L+JP)^{-1}(G+JP)(x)$$

Since K_0 is convex, H is well defined and, by Remarks 3 and 4, H is compact on $\overline{U}\times[0,1]$. In addition, $x \neq H(x,\lambda)$ for all $x \in \partial U$ and $\lambda \in [0,1]$. To see this, suppose the contrary, that is $x = H(x,\lambda)$ for some $x \in \partial U$ and $\lambda \in [0,1]$. If $\lambda = 0$, we should have $x = x_0$, a contradiction since $x_0 \in U$. If $\lambda = 1$, then $Lx = G(x)$ or, equivalently, $Lx = Lx_0 + F_0(x)$. This means that $L(x-x_0)=0$ and $F_0(x)=0$, that is $x \in (x_0 + X_1)\cap \partial U$ and $F_0(x)=0$, which contradicts (3). If $0<\lambda<1$, then $x - x_0 = \lambda[H(x,1) - x_0]$, or

$$(L + JP)(x - x_0) = \lambda[(G + JP)(x) - (L + JP)x_0] = \lambda F_0(x) + \lambda JP(x - x_0).$$

This yields $L(x-x_0)=0$, that is $x \in (x_0+X_1)\cap\partial U$, and $\lambda F_0(x) = (1-\lambda)J(x-x_0)$. Consequently, $\lambda\langle F_0(x), J(x-x_0)\rangle = (1-\lambda)|J(x-x_0)|^2 > 0$, which contradicts (4). Thus our claim is proved. Finally, by the classical topological transversality theory, since $H_0 \equiv x_0$ is essential, we get that H_1 is essential too. Thus, there is $x \in U$ with $x = H_1(x)$, that is $Lx = G(x)$.

Remarks 7. If

$$Lx_0 - JP(\overline{U} - x_0) \subset K,$$

then the map $F_0(x) = -JP(x-x_0)$ satisfies (2)–(4). Therefore, if K_0 is convex and (1), (5) hold, then the map $Lx_0 - JP(x-x_0)$ is essential in M.

8. In case that $x_0 \in U \subset K_0 = K \subset X = Y$ and $L = I$, conditions (1) and (5) are satisfied and $Lx_0 - JP(x-x_0)$ is just the constant map x_0.

9. Suppose $F:\overline{U}\to Y$ is L-compact. Then $QF:\overline{U}\to Y_1$ is also L-compact. This follows by $(L+JP)^{-1}QF = P(L+JP)^{-1}F$. Thus, in Proposition 1, we can set $F_0 = QF$, where $F:\overline{U}\to Y$ is any L-compact map.

Concerning the solvability of the equation $Lx = N(x)$ we have

THEOREM 2. *Assume K_0 is convex and* (1) *holds. Let $x_0 \in U$ and let $N:\overline{U}\to K$ is L-compact. In addition suppose*

$$Lx_0 + QN(\overline{U}) \subset K, \tag{6}$$

$$QN(x) \neq 0 \text{ for any } x \in (x_0 + X_1)\cap\partial U, \tag{7}$$

$$\langle QN(x), J(x - x_0)\rangle \leq 0 \text{ for any } x \in (x_0 + X_1)\cap\partial U, \tag{8}$$

$$Lx \neq (1-\lambda)Lx_0 + \lambda N(x) \text{ for all } x \in \partial U \text{ and } \lambda\in]0,1]. \tag{9}$$

Then there exists $x \in U$ such that $Lx = N(x)$.

Proof. By Proposition 1 and Remark 9, the map $Lx_0 + QN$ is essential in M. Now suppose the conclusion would be false, that is

$$Lx \neq N(x) \text{ for any } x \in U. \tag{10}$$

Consider $H:\overline{U}\times[0,1]\to K, H(x,\lambda)=(1-\lambda)[Lx_0+QN(x)]+\lambda N(x)$, which is well defined by (6) and $N(\overline{U})\subset K$. If $Lx\neq H(x,\lambda)$ for all $x\in\partial U$ and $\lambda\in[0,1]$, then since $H_0=Lx_0+QN$ is essential in M, by Theorem 1, it would follow that $H_1=N$ is essential in M, which contradicts (10). Thus, there exist $x\in\partial U$ and $\lambda\in[0,1]$ such that $Lx=H(x,\lambda)$. Clearly, by (10), $\lambda>0$. Next $Lx=H(x,\lambda)$ is equivalent to the following system

$$Lx=(1-\lambda)Lx_0+\lambda[N(x)-QN(x)],\quad QN(x)=0.$$

Hence $Lx=(1-\lambda)Lx_0+\lambda N(x)$ for some $x\in\partial U$ and $\lambda\in]0,1]$, which contradicts (9). Thus Theorem 1 is proved.

Remark **10.** It is clear that, under the hypotheses of Theorem 2, for each L-compact map $G:\overline{U}\to K$ satisfying $Lx_0+QG(\overline{U})\subset K$ and $G|_{\partial U}=N|_{\partial U}$, there exists $x\in U$ with $Lx=G(x)$. Thus, if in Theorem 2, $K=Y$, then it follows that N is even essential in M.

3. Continuation principles for families of operators having different domains

Let $\mathcal{U}\subset K_0\times[0,1]$ be a nonempty open bounded subset of $K_0\times[0,1]$ and $H:\overline{\mathcal{U}}\to K$ a map on $\overline{\mathcal{U}}$. If $\mathcal{V}$ is any subset of $X\times[0,1]$, we write $\mathcal{V}_\lambda=\{x\in X;(x,\lambda)\in\mathcal{V}\}$ for each $\lambda\in[0,1]$. Denote $H_\lambda:\overline{\mathcal{U}_\lambda}\to K, H_\lambda(x)=H(x,\lambda)$. In this section we study the family $\{H_\lambda;\lambda\in[0,1]\}$ of operators H_λ with different domains $\overline{\mathcal{U}_\lambda}$. The main idea is to reduce the study of this family to that of a certain family $\{\mathcal{H}_\mu;\mu\in[0,1]\}$ of operators from the same domain $\overline{\mathcal{U}}$ into $K\times[0,1]$. Thus, we pass from maps acting between spaces X and Y, to maps acting between the product spaces $X\times\mathbf{R}$ and $Y\times\mathbf{R}$.

Let us denote $\mathcal{X}=X\times\mathbf{R}, \mathcal{Y}=Y\times\mathbf{R}, \mathcal{K}_0=K_0\times[0,1], \mathcal{K}=K\times[0,1]$ and $\mathcal{L}:D(\mathcal{L})\subset\mathcal{X}\to\mathcal{Y}$, where $D(\mathcal{L})=D(L)\times\mathbf{R}$, and $\mathcal{L}(x,\lambda)=(Lx,\lambda)$. It is easy to check that $\mathcal{L}$ is a linear Fredholm map of index zero and

$$\ker\mathcal{L}=X_1\times\{0\},\ \operatorname{Im}\mathcal{L}=Y_2\times\mathbf{R}.$$

Further, let us consider

$$\mathcal{P}:\mathcal{X}\to X_1\times\{0\},\ \mathcal{P}(x,\lambda)=(Px,0),$$
$$\mathcal{Q}:\mathcal{Y}\to Y_1\times\{0\},\ \mathcal{Q}(x,\lambda)=(Qx,0),$$

and

$$\mathcal{J}:X_1\times\{0\}\to Y_1\times\{0\},\ \mathcal{J}(x,0)=(Jx,0).$$

Notice that

(11) $$(\mathcal{L} + \mathcal{JP})^{-1}(y,\lambda) = \left((L + JP)^{-1}y, \lambda\right) \text{ for any } (y,\lambda) \in \mathcal{Y}.$$

Let

$$\mathcal{M} = \left\{\mathcal{F}:\overline{\mathcal{U}} \to \mathcal{K}; \mathcal{F} \text{ is } \mathcal{L}\text{-compact and } \mathcal{L}(x,\lambda) \neq \mathcal{F}(x,\lambda) \text{ on } \partial\mathcal{U}\right\}.$$

THEOREM 3. *Assume $H:\overline{\mathcal{U}} \to K$ is L-compact on $\overline{\mathcal{U}}$ and*

(c) $L(x) \neq H(x,\lambda)$ *for any* $(x,\lambda) \in \partial\mathcal{U}$,

(d) $\mathcal{H}_0:\overline{\mathcal{U}} \to \mathcal{K}, \mathcal{H}_0(x,\lambda) = (H(x,\lambda),0)$ *is essential in* $\mathcal{M}$.

Then there exists $x \in \mathcal{U}_1$ such that $Lx=H(x, 1)$. Moreover, the map $\mathcal{H}_1(x,\lambda) = (H(x,\lambda),1)$ is essential in $\mathcal{M}$.

Proof. Apply Theorem 1 with $\mathcal{X}, \mathcal{Y}, \mathcal{K}_0, \mathcal{K}, \mathcal{U}, \mathcal{L}, \mathcal{M}$ and $\mathcal{H}$ instead of X, Y, K_0, K, U, L, M and H, where

$$\mathcal{H}:\overline{\mathcal{U}} \times [0,1] \to \mathcal{K}, \; \mathcal{H}(x,\lambda,\mu) = (H(x,\lambda),\mu).$$

The map $\mathcal{H}$ is $\mathcal{L}$-compact in virtue of (11).

Remarks **11.** In case that $\mathcal{U}$ has the form $\mathcal{U} = U \times [0, 1]$, condition (d) implies (b) (use a similar argument with that in [7, Remark 1]).

12. For $X=Y$, $K_0=K$ and $L=I$, Theorem 3 becomes Proposition 2 in [7].

The next result is concerned with a sufficient condition for that (d) holds, namely that H_0 be homotopic on $\mathcal{U}_0$ with a map of the form $Lx_0+F_0(x)$, like that in Proposition 1.

THEOREM 4. *Suppose that K_0 is convex and*

(12) $$(L + JP)^{-1}(K + JP(K_0)) \subset K_0.$$

Let $F_0:K_0 \to Y_1$ be L-completely continuous on K_0, $x_0 \in \mathcal{U}_0$ and the following conditions hold

(13) $$Lx_0 + F_0(K_0) \subset K,$$

(14) $$F_0(x) \neq 0 \;\; \text{for any } x \in (x_0 + X_1) \cap \partial\mathcal{U}_0,$$

(15) $$\langle F_0(x), J(x - x_0)\rangle \leq 0 \;\; \text{for any } x \in (x_0 + X_1) \cap \partial\mathcal{U}_0.$$

If $H:\overline{\mathcal{U}} \to K$ is L-compact, satisfies (c) *and*

(16) $$Lx \neq (1 - \mu)(Lx_0 + F_0(x)) + \mu H(x,0) \;\; \text{for } (x,0) \in \partial\mathcal{U}, \mu \in]0,1[,$$

then there exists $x \in \mathcal{U}_1$ with $Lx = H(x,1)$.

Proof. We show that condition (d) is satisfied and then we apply Theorem 3. For this, let us define

$$\widetilde{\mathcal{H}}:\overline{\mathcal{U}}\times[0,1]\to\mathcal{K},\ \widetilde{\mathcal{H}}(x,\lambda,\mu)=\left((1-\mu)(Lx_0+F_0(x))+\mu H(x,\lambda),0\right).$$

We have that $\widetilde{\mathcal{H}}$ is $\mathcal{L}$-compact on $\overline{\mathcal{U}}\times[0,1]$. On the other hand,

$$\mathcal{L}(x,\lambda)\neq\widetilde{\mathcal{H}}(x,\lambda,\mu)\ \text{for}\ (x,\lambda)\in\partial\mathcal{U}\ \text{and}\ \mu\in[0,1]$$

(use (16) in case that $\mu\in]0,1[$, (14) when $\mu=0$, and (c) for $\mu=1$).

For $\mu=0,\widetilde{\mathcal{H}}_0(x,\lambda)=(Lx_0+F_0(x),0)=\mathcal{L}(x_0,0)+\mathcal{F}_0$, where $\mathcal{F}_0:\overline{\mathcal{U}}\to Y_1\times\{0\}$, $\mathcal{F}_0(x,\lambda)=(F_0(x),0)$. Now we can easily check that all hypotheses of Proposition 1 are satisfied for

$$\mathcal{X},\mathcal{Y},\mathcal{K}_0,\mathcal{K},\mathcal{U},\mathcal{L},\mathcal{J},\mathcal{P},\mathcal{F}_0,\mathcal{M}\ \text{and}\ (x_0,0)$$

instead

$$X,Y,K_0,K,U,L,J,P,F_0,M\ \text{and}\ x_0.$$

It follows that $\widetilde{\mathcal{H}}_0$ is essential in $\mathcal{M}$ and so, by Theorem 1, $\widetilde{\mathcal{H}}_1=\mathcal{H}_0$ is essential in $\mathcal{M}$ too. Thus, (d) holds and Theorem 3 applies.

Remarks **13.** For $K_0=X$ and $K=Y$ conditions (12), (13) hold.

14. The map $F_0(x)=-JP(x-x_0)$ satisfies (14) and (15).

15. For $X=Y$, $K_0=K$ and $L=I$, Theorem 4 becomes Corollary 1 in [7] (in that case $F_0=0$).

The last result of this section concerns the equation $Lx=N(x)$, being of the type of Theorem 2.

THEOREM 5. *Assume K_0 is convex and* (12) *holds. Let $x_0\in\mathcal{U}_0$ and let $N:K_0\to K$ be L-completely continuous. In addition suppose*

(17) $$QN(x)\neq0\ \textit{for any}\ x\in x_0+X_1\ \textit{with}\ (x,0)\in\partial\mathcal{U},$$

(18) $$\langle QN(x),J(x-x_0)\rangle\leq0\ \textit{for any}\ x\in x_0+X_1\ \textit{with}\ (x,0)\in\partial\mathcal{U},$$

(19) $$Lx\neq(1-\lambda)Lx_0+\lambda N(x)\ \textit{for any}\ (x,\lambda)\in\partial\mathcal{U}\ \textit{with}\ \lambda\in]0,1].$$

Then there exists $x\in\mathcal{U}_1$ with $Lx=N(x)$.

Proof. Check that all the assumptions of Theorem 2 are satisfied for

$$\mathcal{X},\mathcal{Y},\mathcal{K}_0,\mathcal{K},\mathcal{U},\mathcal{L},\mathcal{J},\mathcal{P},\mathcal{Q},\mathcal{N}\ \text{and}\ (x_0,0)$$

instead of

$$X,Y,K_0,K,U,L,J,P,Q,N\ \textit{and}\ x_0,$$

where $\mathcal{N}:\overline{\mathcal{U}}\to\mathcal{K},\mathcal{N}(x,\lambda)=(N(x),1)$. Then apply Theorem 2.

4. No degree versions of some continuation theorems of Capietto-Mawhin-Zanolin

To make the results of Section 3 useful we need to get methods for the construction of a set $\mathcal{U} \subset K_0 \times [0,1]$ with the desired properties. Such a method was described in [1], in the frame of the coincidence degree theory. In this section we give a no degree approach to that method.

Suppose K_0 is convex, $x_0 \in K_0, F_0 : K_0 \to Y_1$ is L-completely continuous and (12), (13) hold. In addition assume

(20) $$F_0(x) \neq 0 \text{ for any } x \in x_0 + X_1 \text{ with } x \neq x_0$$

(21) $$\langle F_0(x), J(x - x_0) \rangle \leq 0 \text{ for any } x \in x_0 + X_1.$$

Let $H: K_0 \times [0,1] \to K$ be L-completely continuous and denote

$$\mathcal{S} = \{(x, \lambda) \in K_0 \times [0,1];\ Lx = H(x, \lambda)\},$$

$$\mathcal{S}(x_0) = \{(x,0); x \in K_0 \text{ and } Lx = (1-\mu)(Lx_0 + F_0(x)) + \mu H(x,0) \text{ for some } \mu \in [0,1]\}.$$

Also consider a continuous functional $\Phi : K_0 \times [0,1] \to \mathbf{R}$.

THEOREM 6. *Assume there are constants c_- and c_+, $c_- < c_+$, such that if we denote $\mathcal{V} = \Phi^{-1}(]c_-, c_+[)$ the following conditions are satisfied:*

(i1) $\mathcal{S} \cap \mathcal{V}$ *is bounded,*

(i2) $\Phi(\mathcal{S}) \cap \{c_-, c_+\} = \varnothing,$

(i3) $\mathcal{S}(x_0)$ *is bounded and included by* $\mathcal{V}$.

Then there exists $x \in \mathcal{V}_1$ *with* $Lx = H(x, 1)$.

Proof. Denote $\mathcal{S}^* = \Phi^{-1}([c_-, c_+]) \cap \mathcal{S}$. By (i2), $\mathcal{S}^* = \mathcal{V} \cap \mathcal{S}$, while, by (i1) and the continuity of $\Phi, \mathcal{S}^*$ is compact. Hence $\mathcal{S}^*$ is a compact set included by the open set $\mathcal{V}$. Thus, there exists a bounded open set $\mathcal{U}'$ of $\mathcal{K}_0$ with

$$\mathcal{S}^* \subset \mathcal{U}' \subset \overline{\mathcal{U}}' \subset \mathcal{V}$$

On the other hand, by (i3), $\mathcal{S}(x_0)$ is another compact set included by $\mathcal{V}$. Thus, there exists a bounded open set $\mathcal{U}''$ of $\mathcal{K}_0$ with

$$\mathcal{S}(x_0) \subset \mathcal{U}'' \subset \overline{\mathcal{U}}'' \subset \mathcal{V}$$

Now the conclusion follows by Theorem 4 for $\mathcal{U} = \mathcal{U}' \cup \mathcal{U}''$.

Recall that the functional Φ is said to be *proper* on $\mathcal{S}$ provided that $\mathcal{S} \cap \Phi^{-1}(]a, b[)$ is bounded (equivalently, relatively compact) for each bounded real interval $]a, b[$.

COROLLARY 1. *Suppose*

(i1') Φ *is proper on* $\mathcal{S}$, (i2') *Φ is lower bounded on $\mathcal{S}$ and there is a sequence (c_j) of real numbers with $c_j \to \infty$ and $c_j \notin \Phi(\mathcal{S})$ for all j,*

(i3') *$\mathcal{S}(x_0)$ is bounded.*

Then there exists $x \in K_0$ such that $Lx = H(x, 1)$.

Proof. By (i3') and the L-complete continuity of F_0 and H, we have that $\mathcal{S}(x_0)$ is in fact compact. Since Φ is continuous, there are constants a and b such that $a < \Phi(x, \lambda) < b$ for any $(x, \lambda) \in \mathcal{S}(x_0)$. Further, by using (i2'), we can choose c_- and j sufficiently large that

$$c_- \leq a, \quad c_- < \inf\{\Phi(x,\lambda); (x,\lambda) \in \mathcal{S}\}, \quad c_+ = c_j \geq b.$$

Now we easily check (i1)–(i3) and we apply Theorem 6.

Remark **16.** For $X=Y$, $K_0=K$ and $L=I$, the results of this section reduce to Theorem 1 and Corollary 2 from [7].

Applications of Corollary 1 will be given in a forthcoming paper [8].

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