BULGARIAN ACADEMY OF SCIENCES

Mathematica Balkanica

Editor-in-chief: Blagovest Sendov

ON A CLASS OF LINEAR APPROXIMATING OPERATORS

Octavian AGRATINI

New Series Vol. 11, 1997, Fasc. 3 - 4

Mathematica
Balkanica

New Series Vol.11, 1997, Fasc. 3-4

On a Class of Linear Approximating Operators

O. Agratini

Presented by Bl. Sendov

In this paper we introduce a new class of linear approximating operators $(L_{nr})_{n\geq 1}$, $r = 0, 1, 2, \ldots$, for the functions $f \in C^r[0,1]$. In order to construct them we use Taylor's polynom of r degree and a classical class of linear positive operators generated by a probabilistic method. Also, we study approximation degree with the modulus of continuity of first and second order. $(L_{nr}f)_{n\geq 1}$ include as a particular case the generalized Bernstein polynomials defined by G.H. Kirov in [5].

1. Introduction

Any problem concerning the consideration of positive linear approximating operators, on the space of continuous real-valued functions on an interval of the real axis can naturally be interpreted as a problem of probability theory. There are several papers dedicated to this subject.

Let $(X_n)_{n\geq 1}$ be a real sequence of random variables having the following distribution: $P(X_n = x_{nk}) = p_{nk}(x)$ where $x_{nk} \in [0,1]$, $0 \leq x \leq 1$ represents a parameter, $k \in I \subseteq \mathbb{N}$.

Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function. We consider an operator

$$L_n : C[0,1] \longrightarrow C[0,1], \quad f \longrightarrow L_n f,$$

defined by the following formula:

$$(L_n f)(x) = \sum_{k\in I} p_{nk}(x) f(x_{nk}). \tag{1}$$

This type of operator has been considered in remarkable papers by D.D. Stancu [7],[8] and W. Feller [2] and borrowed by many other authors either to get through a probabilistic way to some classical linear positive operators

(Bernstein, Mirakyan-Favard-Szász, Baskakov, Meyer-König and Zeller) or to get some important generalizations of these operators.

In our previous paper [1] we approached a similar formula for functions of two variables. It is evident that the operator defined by the formula (1) is a positive linear operator and according to the definition of a distribution we have

$$(L_n e_o)(x) = \sum_{k\in I} p_{nk}(x) = 1. \tag{2}$$

We denote $e_j : [0,1] \longrightarrow \mathbb{R}$, $e_j(x) = x^j$, $j = 0, 1, 2, \ldots$

Starting from (1), the purpose of this paper is to form a new sequence of linear operators. We will also evaluate the order of approximation by means of the modulus of continuity of first and second order. As a particular case, a class of the Bernstein polynomials introduced in [5] are to be found.

2. Construction of the operators

By $C^r[0,1]$, $r = 0, 1, 2, \ldots$, we denote the set of all functions $f : [0,1] \longrightarrow \mathbb{R}$ with a continuous derivative of order r on [0,1].

Let $f \in C^r[0,1]$ and $(T_r f)(x_{nk}; \bullet)$ be Taylor's polynomial of r degree associated to the function f on the point x_{nk}:

$$(T_r f)(x_{nk}; x) = \sum_{i=0}^{r} \frac{f^{(i)}(x_{nk})}{i!}(x - x_{nk})^i. \tag{3}$$

We define the operators:

$$L_{nr} : C^r[0,1] \longrightarrow C[0,1], \quad f \longrightarrow L_{nr} f$$

where

$$(L_{nr} f)(x) = \sum_{k\in I} (T_r f)(x_{nk}; x) p_{nk}(x). \tag{4}$$

It is easy to verify that these operators are linear and choosing $r = 0$, we obtain the operators presented in (1) ($L_{no} f = L_n f$).

Further on, we consider $r \geq 1$ integer.

3. Approximation degree

First, we need Taylor's modified formula with the rest under integral form:

$$f(x) = (T_r f)(x_{nk}; x) \tag{5}$$

$$+\frac{(x-x_{nk})^r}{(r-1)!}\int_0^1(1-t)^{r-1}(f^{(r)}(x_{nk}+t(x-x_{nk}))-f^{(r)}(x_{nk}))dt.$$

Second, we need the modulus of continuity of function f defined by:

(6) $$\omega_1(f;\delta)=\sup\{|f(x)-f(y)|:\ x,y\in[0,1],\ |x-y|\le\delta\},\ 0<\delta\le 1.$$

From relations (2),(5),(6) successively applied, for any $x\in[0,1]$ we obtain:

$$|f(x)-(L_{nr}f)(x)|=\left|\sum_{k\in I}f(x)p_{nk}(x)-\sum_{k\in I}\sum_{i=0}^{r}\frac{f^{(i)}(x_{nk})}{i!}(x-x_{nk})^i p_{nk}(x)\right|$$

$$=\left|\sum_{k\in I}\frac{(x-x_{nk})^r}{(r-1)!}p_{nk}(x)\int_0^1(1-t)^{r-1}(f^{(r)}(x_{nk}+t(x-x_{nk}))-f^{(r)}(x_{nk}))dt\right|$$

$$\le\sum_{k\in I}\frac{|x-x_{nk}|^r}{(r-1)!}p_{nk}(x)\int_0^1(1-t)^{r-1}\omega_1(f^{(r)};t|x-x_{nk}|)dt.$$

Using the following known property of the modulus of continuity

$$\omega_1(f;\lambda\delta)\le(\lambda+1)\omega_1(f;\delta),\quad \lambda>0,$$

with $\lambda=t|x-x_{nk}|n^{1/2}$ and $\delta=n^{-1/2}$, we can write further:

$$|f(x)-(L_{nr}f)(x)|$$

$$\le\sum_{k\in I}\frac{|x-x_{nk}|^r}{(r-1)!}p_{nk}(x)\int_0^1(1-t)^{r-1}(t|x-x_{nk}|n^{1/2}+1)\omega_1(f^{(r)};n^{-1/2})dt$$

$$=\sum_{k\in I}\frac{|x-x_{nk}|^r}{(r-1)!}p_{nk}(x)\omega_1(f^{(r)};n^{-1/2})\left\{|x-x_{nk}|n^{1/2}\int_0^1 t(1-t)^{r-1}dt\right.$$

$$\left.+\int_0^1(1-t)^{r-1}dt\right\}=\sum_{k\in I}\frac{|x-x_{nk}|^r}{(r+1)!}p_{nk}(x)\omega_1(f^{(r)};n^{-1/2})\left(|x-x_{nk}|\sqrt{n}+r+1\right)$$

$$\le\frac{\sqrt{n}+r+1}{(r+1)!}\sum_{k\in I}|x-x_{nk}|^r p_{nk}(x)\omega_1\left(f^{(r)};\frac{1}{\sqrt{n}}\right),$$

because $|x-x_{nk}|\le 1$.

Applying the Cauchy-Schwarz inequality and taking into account the identity (2), we get:

(7) $$\sum_{k\in I}|x-x_{nk}|^r p_{nk}(x)\le\left(\sum_{k\in I}(x-x_{nk})^{2r}p_{nk}(x)\right)^{1/2}.$$

Definition. ([3], p. 189) The expectation of the variable $(Y-a)^k$ is called the moment of the k-th order about a of the random variable Y:

$$\nu_k(Y;a) = E(Y-a)^k.$$

Consequently (7) can be rewritten:

$$\sum_{k\in I} |x - x_{nk}|^r p_{nk}(x) \le \nu_{2r}^{1/2}(X_n;x)$$

and we find

$$|f(x) - (L_{nr}f)(x)| \le \frac{\sqrt{n}+r+1}{(r+1)!}\nu_{2r}^{1/2}(X_n;x)\omega_1\left(f^{(r)};\frac{1}{\sqrt{n}}\right).$$

Now we can state the following proposition:

Theorem. *Let be the sequences $(L_n)_{n\ge1}$ and $(L_{nr})_{n\ge1}$ defined in (1) respectively in (4). If the following constants $\lambda_r > 0$ and $\alpha \ge \frac{1}{2}$ exist with the property:*

$$\nu_{2r}(X_n;x) \le \lambda_r n^{-2\alpha},$$

for any natural number n, then the inequality

$$|f(x) - (L_{nr}f)(x)| \le \frac{\lambda_r^{1/2}(\sqrt{n}+r+1)}{(r+1)!n^\alpha}\omega_1\left(f^{(r)};\frac{1}{\sqrt{n}}\right) \tag{8}$$

holds.

R e m a r k. The above result implies:

$$||f - L_{nr}f|| = \mathbf{0}\left(n^{\frac{1}{2}-\alpha}\omega_1\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\right), \tag{9}$$

where $||\bullet||$ stands for the uniform norm.

It is known that besides ω_1, the modulus of smoothness ω_2 is frequently used in quantitative approximation. This ω_2 is defined, for $f \in C[a,b]$, by

$$\omega_2(f;\delta) = \sup\{|f(x-h)-2f(x)+f(x+h)| : x, x\pm h \in [a,b],\ 0 < h \le \delta\}, 0 < \delta \le b-a.$$

H.H. Gonska [4] established the following inequality:

$$\omega_1(f;\delta) \le \left(3 + \frac{2(b-a)}{\delta}\right)\omega_2(f;\delta) + \frac{6\delta}{b-a}||f||,$$

for all $f \in C[a,b]$ and $0 < \delta \leq b - a$.

Our result (8) becomes:

$$|f(x) - (L_{nr}f)(x)| \leq c_r(n)\omega_2\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) + d_r(n)||f^{(r)}||,$$

where:

$$c_r(n) = \frac{\lambda_r^{1/2}}{(r+1)!}\left(2n + (2r+5)\sqrt{n} + 3(r+1)\right) n^{-\alpha}$$

and

$$d_r(n) = \frac{6\lambda_r^{1/2}}{(r+1)!}\left(1 + \frac{r+1}{\sqrt{n}}\right) n^{-\alpha}.$$

4. Example

We choose $I = \{0, 1, 2, \ldots, n\}$, $x_{nk} = \frac{k}{n}$, $p_{nk}(x) = \binom{n}{k} x^k(1-x)^{n-k}$. In this case $L_n f$ coincides with the classical Bernstein polynomial $B_n f$ and $L_{nr} f$ becomes the generalized Bernstein polynomial of the (n,r)-th order for a function $f \in C^r[0,1]$, introduced and studied in the recent paper [5]. For any $x \in [0,1]$ we have:

$$\sum_{k \in I}(x - x_{nk})^{2r} p_{nk} = n^{-2r}\sum_{k=0}^{n}(k - nx)^{2r}\binom{n}{k} x^k(1-x)^{n-k} \leq An^{-r},$$

where A is a constant depending only on r (see e.g. [6], p.248).

If we take $\lambda_r = A$ and $\alpha = \frac{r}{2} \geq \frac{1}{2}$ the condition of the theorem given in Section 3 is satisfied and we have:

$$|f(x) - (B_n f)(x)| \leq \frac{A^{1/2}(\sqrt{n} + r + 1)}{(r+1)!\, n^{r/2}}\omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

Consequently we can write

$$||f - B_n f|| = 0\left(n^{\frac{1-r}{2}}\omega_1\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\right).$$

This result is similar with the one obtained by G.H. Kirov in the paper [5] published in this journal.

References

[1] O. A g r a t i n i, On the construction of approximating linear positive operators by probabilistic methods. *Studia Univ. Babeş-Bolyai* **38**, No 4 (1993), 45-50.

[2] W. F e l l e r, *An Introduction to Probability Theory and Its Applications.* New York-London, 1957.

[3] B. V. G n e d e n k o, *The Theory of Probability.* Mir, Moscow, 1969.

[4] H. H. G o n s k a, Quantitative Korovkin type theorems on simultaneous approximation. *Math. Zeitschr.* **186** (1984), 419-433.

[5] G. H. K i r o v, A generalization of the Bernstein polynomials. *Mathematica Balkanica* **6**, No 2 (1992), 147-153.

[6] I. P. N a t a n s o n, *Constructive Function Theory* (In Russian). Moscow-Leningrad, 1949.

[7] D. D. S t a n c u, Use of probabilistic methods in the theory of uniform approximation of continuous functions. *Rev. Roum. Math. Pures Appl.* **14**, No 5 (1969), 673-691.

[8] D. D. S t a n c u. Probabilistic methods in the theory of approximation of functions of several variables by linear positive operators. In: *Approximation Theory* (ed. A. Talbot), London-New York, 1970, 329-342.

"Babeş-Bolyai" University
Faculty of Mathematics and Informatics
M.Kogălniceanu 1
3400 Cluj-Napoca
ROMANIA

Received: 18.06.1996