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ON SIMULTANEOUS APPROXIMATION BY STANCU-BERNSTEIN OPERATORS

OCTAVIAN AGRATINI[(1)]

1. INTRODUCTION

By using a probabilistic method D.D.Stancu [4] constructed a linear positive polynomial operator $L_{m,r}^{\alpha,\beta}$ of Bernstein type, depending on a non negative integer parameter r $(2r < m)$ and on two real parameters α and β such as $0 \le \alpha \le \beta$. The expression of this operator is presented below:

$$\left(L_{m,r}^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{m} w_{m,k,r}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1}$$

where $f \in \mathrm{C}[0,1]$ and

$$w_{m,k,r}(x) = \begin{cases} \binom{m-r}{k} x^k (1-x)^{m-r-k+1}, & 0 \le k < r \\ \binom{m-r}{k} x^k (1-x)^{m-r-k+1} + \\ +\binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k}, & r \le k \le m-r \\ \binom{m-r}{k-r} x^{k-r+1} (1-x)^{m-k}, & m-r < k \le m \end{cases}$$

It is obvious that for $\alpha = \beta = 0$ and $r = 0$ or $r = 1$, the operator becomes the well-known Bernstein operator. We mention that $L_{m,2}^{0,0}$ has been given earlier by H.Brass.

Furthermore, in the same paper, the author was able to express the operator by means of the fundamental Bernstein polynomials in the following form:

$$\left(L_{m,r}^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{m-r} p_{m-r,k}(x) \left[(1-x) f\left(\frac{k+\alpha}{m+\beta}\right) + x f\left(\frac{k+r+\alpha}{m+\beta}\right)\right] \tag{2}$$

where

$$p_{m-r,k}(x) = \binom{m-r}{k} x^k (1-x)^{m-r-k}.$$

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For our exposure we need also an old result obtained by D.D.Stancu [3]. He studied a class of Bernstein operators depending on two real parameters $0 \le a \le b$. This operator is defined as follows:

$$\left(S_n^{a,b} f\right)(x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k+a}{n+b}\right), \tag{3}$$

where $f \in \mathbf{C}\,[0,1]$.

The aim of our lecture is to prove an estimation for the difference

$$\left|\left(L_{m,r}^{\alpha,\beta} f\right)^{(s)}(x) - f^{(s)}(x)\right|, \quad s \le m-r,$$

which involves the first order modulus of continuity ω_1 of s-th and (s+1)-th derivative of f, where $f \in C^{(s+1)}[0,1]$. The technique of evaluation follows a classical way presented in numerous papers, such as [2].

2. Preliminary results

At first, we differentiate the relation (3) s times and we get (see [5]):

$$\left(S_n^{a,b} f\right)^{(s)}(x) = n(n-1)\ldots(n-s+1)\sum_{k=0}^{n-s}\binom{n-s}{k}x^k(1-x)^{n-k-s}\Delta^s_{\frac{1}{n+b}} f\left(\frac{k+a}{n+b}\right), \tag{4}$$

where $\Delta^s_{\frac{1}{n+b}} f\left(\frac{k+a}{n+b}\right)$ represents the difference of order s $(s<n)$ on the function f with the step $\frac{1}{n+b}$ starting from the value $\frac{k+a}{n+b}$.

By using the mean value theorem, we can write:

$$\Delta^s_{\frac{1}{n+b}} f\left(\frac{k+a}{n+b}\right) = \frac{1}{(n+b)^s} f^{(s)}\left(\frac{k+a+s\theta_k}{n+b}\right), \quad \theta_k \in (0,1). \tag{5}$$

Clearly:

$$f^{(s)}\left(\frac{k+a+s\theta_k}{n+b}\right) = f^{(s)}(x) - \left(x - \frac{k+a+s\theta_k}{n+b}\right) f^{(s+1)}(x) - $$
$$- \int_x^{\frac{k+a+s\theta_k}{n+b}} \left(f^{(s+1)}(x) - f^{(s+1)}(t)\right) dt. \tag{6}$$

If we note

$$\frac{n(n-1)\ldots(n-s+1)}{(n+b)^s} = \alpha_{n,s}^b \tag{7}$$

and substitute (6) in (5) and (5) in (4) we obtain:

$$\left(S_n^{a,b} f\right)^{(s)}(x) = \alpha_{n,s}^b \sum_{k=0}^{n-s}\left\{ f^{(s)}(x) - \left(x - \frac{k+a+s\theta_k}{n+b}\right) f^{(s+1)}(x) - \right.$$
$$\left. - \int_x^{\frac{k+a+s\theta_k}{n+b}} \left(f^{(s+1)}(x) - f^{(s+1)}(t)\right) dt \right\} \binom{n-s}{k} x^k (1-x)^{n-k-s} =$$

$$= \alpha_{n,s}^b f^{(s)}(x) - \alpha_{n,s}^b \sum_{k=0}^{n-s} \binom{n-s}{k} \left(x - \tfrac{k+a}{n+b}\right) x^k (1-x)^{n-k-s} f^{(s+1)}(x) +$$

$$+\alpha_{n,s}^b \sum_{k=0}^{n-s} \tfrac{s\theta_k}{n+b} f^{(s+1)}(x) \binom{n-s}{k} x^k (1-x)^{n-k-s} -$$

$$-\alpha_{n,s}^b \sum_{k=0}^{n-s} \binom{n-s}{k} x^k (1-x)^{n-k-s} \int_x^{\frac{k+a+s\theta_k}{n+b}} \left(f^{(s+1)}(x) - f^{(s+1)}(t)\right) dt. \tag{8}$$

But

$$|f^{(s+1)}(x) - f^{(s+1)}(t)| \le (1 + |x-t|\delta^{-1})\omega_1(f^{(s+1)};\delta),$$

here ω_1 is defined by

$$\omega_1(f,\delta) = \sup_{|x'-x''|\le\delta} |f(x') - f(x'')|$$

, x'' being points from [0,1] and δ a positive number.

We can write successively:

$$\int_x^{\frac{k+a+s\theta_k}{n+b}} |f^{(s+1)}(x) - f^{(s+1)}(t)| dt \le$$

$$\le \left(\left| \tfrac{k+a+s\theta_k}{n+b} - x \right| + \tfrac{1}{2\delta} \left(\tfrac{k+a+s\theta_k}{n+b} - x \right)^2 \right) \omega_1(f^{(s+1)};\delta) \le$$

$$\le \left\{ \left(1 + \tfrac{s}{\delta(n+b)}\right) \left| \tfrac{k+a}{n+b} - x \right| + \tfrac{s}{n+b} + \tfrac{s^2}{2(n+b)^2\delta} + \tfrac{1}{2\delta} \left(\tfrac{k+a}{n+b} - x \right)^2 \right\} \omega_1(f^{(s+1)};\delta).$$

Substituting this result in (8) and taking down $\varphi_x(t) = |t-x|$, $0 \le t \le 1$, we can ntinue with the following increases:

$$|\left(S_n^{a,b} f\right)^{(s)}(x) - f^{(s)}(x)| \le |\alpha_{n,s}^b - 1||f^{(s)}(x)| + \alpha_{n,s}^b \left(S_{n-s}^{a,b+s}\varphi_x\right)(x)|f^{(s+1)}(x)| +$$

$$+\alpha_{n,s}^b \tfrac{s}{n+b}|f^{(s+1)}(x)| + \alpha_{n,s}^b \left\{ \left(1 + \tfrac{s}{\delta(n+b)}\right) \left(S_{n-s}^{a,b+s}\varphi_x\right)(x) + \tfrac{s}{n+b} + \right.$$

$$\left. + \tfrac{s^2}{2(n+b)^2\delta} + \tfrac{1}{2\delta} \left(S_{n-s}^{a,b+s}\varphi_x^2\right)(x) \right\} \omega_1(f^{(s+1)};\delta) \le |\alpha_{n,s}^b - 1| \|f^{(s)}\| +$$

$$\left(\left(S_{n-s}^{a,b+s}\varphi_x\right)(x) + \tfrac{s}{n+b} \right) \|f^{(s+1)}\| + \left\{ \left(1 + \tfrac{s}{\delta(n+b)}\right) \left(S_{n-s}^{a,b+s}\varphi_x\right)(x) + \tfrac{s}{n+b} + \right.$$

$$\left. + \tfrac{s^2}{2(n+b)^2\delta} + \tfrac{1}{2\delta} \left(S_{n-s}^{a,b+s}\varphi_x^2\right)(x) \right\} \omega_1(f^{(s+1)};\delta). \tag{9}$$

In the relation above we have used $\alpha_{n,s}^b \le 1$; also, we mention that $\|\cdot\|$ is sup-norm on $[0,1]$.

By making use of the Cauchy inequality and according to Stancu (see [3]), we have:

$$\left(S_{n-s}^{a,b+s}\varphi_x\right)(x) \le \left\{ \left(S_{n-s}^{a,b+s} e_0\right)(x) \left(S_{n-s}^{a,b+s}\varphi_x^2\right)(x) \right\}^{1/2} = \left(S_{n-s}^{a,b+s}\varphi_x^2\right)^{1/2}(x) \le$$

Corollary 1. *Under the hypothesis of Theorem 2,*

$$\lim_{m\to\infty} \left(L_{m,r}^{\alpha,\beta} f\right)^{(s)}(x) = f^{(s)}(x),$$

the convergence being uniform on $[0,1]$.

Finally, we mention that in [1] we considered an extension in the sense of Kantorovich of the operators $L_{m,r}^{\alpha,\beta}$. For this extension we established some quantitative theorems representing estimations of the order of approximation.

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