MATHEMATICA, Tome 40(63), N° 1, 1998, pp. 3–8

AN ASYMPTOTIC PROPERTY OF INTEGRAL TYPE OPERATORS

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1 Introduction

Let J be a subinterval of real axis not necessarily bounded and for each integer $n \geq 1$ let $I_n \subset \mathbb{N}$ be a set of indexes. Also, we consider $p_{n,k}$ $(n \geq 1, k \in I_n)$ real non-negative valued functions with the property $p_{n,k} \in C(J)$ and the nodes $x_{n,k} \in J$. Let $(X_n)_{n \geq 1}$ be a sequence of random variables having the following distributions: $P(X_n = x_{n,i}) = p_{n,i}(x)$, $i \in I_n$. By the definition of these distributions we have for any $n \geq 1$,

$$(1) \qquad \sum_{i \in I_n} p_{n,i}(x) = 1, \quad x \in J.$$

Connections between probability and positive linear operators have been discussed in many papers. In this direction, associated to the above random variables, we can consider the well-known class of linear and positive operators defined by

$$(2) \qquad (L_n f)(x) = \sum_{k \in I_n} p_{n,k}(x) f(x_{n,k}),$$

where $f \in C(J)$.

In [6] Durrmeyer introduced a new kind of modified Bernstein polynomial operator on $L_1[0, 1]$, the space of Lebesgue integrable functions on [0, 1]. In order to generalize L_n to an integral operator we follow Durrmeyer and take $p_{n,k}$ as an integral kernel. In this way we define

$$(3) \qquad (D_n f)(x) = \sum_{k \in I_n} c_{n,k} p_{n,k}(x) \int_J p_{n,k}(t) f(t) \mathrm{d}t,$$

where the coefficients $c_{n,k}$ satisfy the normalization property, that means $c_{n,k}^{-1} = \int_J p_{n,k}(t)\mathrm{d}t$, $k \in I_n$, $n \geq 1$. This fact together with (1) guarantees

$$(4) \qquad (D_n e_0)(x) = e_0(x),$$

where, generally, $e_k(x) = x^k$, $x \in J$ and $k \geq 0$.

The operators in question are defined on the space $L_1(J)$ and they include the ones considered in the literature under the name of "modified operators" being the integral analogue of the Bernstein, Baskakov, Meyer-König and Zeller operators, see [5], [7], [3]. For a similar construction, in [1] we indicated sufficient conditions which ensure the uniform convergence of the sequence. If for a certain sequence of operators (D_n) and a certain function f the relation $\lim_{n\to\infty}(D_n f)(x) = f(x)$ holds, the question arises how fast $(D_n f)(x)$ tends to $f(x)$. An answer to this problem is possible to be given in different directions and one of them is that in which f is supposed to be at least twice differentiable in a point $x \in J$. We refer here to the Voronovskaja-type formulae.

The object of this paper is to present such a formula for the operators defined by (3). Of course, this implies that stronger conditions on $p_{n,k}$ must be imposed. Also, we are going to give explicit formulations of our formula for some classical kernels like Bernstein and Baskakov. The last example which is presented here, leads to an unexpected surprise.

2. Main result

Firstly, we define:

$$T_{n,l}(x) = \sum_{k\in I_n} c_{n,k} p_{n,k}(x) \int_J p_{n,k}(t)(t-x)^l \mathrm{d}t, \quad l \geq 0. \tag{5}$$

It is clear that $T_{n,0}(x) = 1$ and $T_{n,l}$ are continuous functions on J for any $l > 0$. In what follows, we will consider the following assumptions:

(i) $0 < \alpha_1 \leq \alpha_2 < \alpha_3$ exist such as

$$T_{n,k}(x) = \mathrm{O}(n^{-\alpha_k}),\ k \in \{1,2\} \quad \text{and} \quad T_{n,l}(x) = \mathrm{O}(n^{-\alpha_3}) \text{ for } l \geq 4; \tag{6}$$

(ii) if J is an unbounded interval then

$$f(t) = \mathrm{O}(t^\beta) \text{ as } t \to \infty, \text{ for some } \beta \geq 2. \tag{7}$$

THEOREM. *Let $f \in L_1(J)$ and be bounded on every compact subinterval of J admitting a derivative of order two at a fixed point $x \in J$. Under the hypotheses (6) and (7) it results*

$$\lim_{n\to\infty} n^\lambda((D_n f)(x) - f(x)) = \varphi(x),$$

where $\lambda = \min\{\alpha_1, \alpha_2\}$ and

$$\varphi(x) = \begin{cases} f'(x)a(x), & \alpha_1 < \alpha_2 \\ f'(x)a(x) + \frac{f''(x)}{2}b(x), & \alpha_1 = \alpha_2. \end{cases}$$

Here $a(x)$ and $b(x)$ are defined as follows:

$$a(x) = \lim_{n\to\infty} T_{n,1}(x)n^{\alpha_1}, \quad b(x) = \lim_{n\to\infty} T_{n,2}(x)n^{\alpha_2}.$$

Proof. By Taylor's expansion of f, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \mu_x(t)(t-x)^2, \tag{8}$$

where $\mu_x(t)$ tends to zero as t tends to x. D_n being a linear operator and taking into account relations (4) and (5), we can write:

$$(D_n f)(x) - f(x) = f'(x)T_{n,1}(x) + \frac{f''(x)}{2}T_{n,2}(x) + (R_n\mu_x)(x),$$

where

$$(R_n\mu_x)(x) = \sum_{k\in I_n} c_{n,k}p_{n,k}(x)\int_J p_{n,k}(t)\mu_x(t)(t-x)^2\mathrm{d}t.$$

In order to prove the theorem it is sufficient to show that

$$\lim_{n\to\infty} n^{\lambda}(R_n\mu_x)(x) = 0. \tag{9}$$

Case 1. J is a bounded interval.

Because $\mu_x(t) \to 0$ as $t \to x$ we deduce: for a given $\varepsilon > 0$ a $\delta > 0$ exists such that $|\mu_x(t)| < \varepsilon$ whenever $0 < |t-x| < \delta$. For any $t \in J$ with the property $|t-x| \geq \delta$ we have $|\mu_x(t)| \leq M(t-x)^2/\delta^2$ because μ_x is bounded by some constant M. We can write $|\mu_x(t)| < \varepsilon + M(t-x)^2/\delta^2$ which implies:

$$|(R_n\mu_x)(x)| < \varepsilon T_{n,2}(x) + M\delta^{-2}T_{n,4}(x).$$

Under the given assumption (6) we get

$$n^{\lambda}|(R_n\mu_x)(x)| < \varepsilon \mathrm{O}(n^{\lambda-\alpha_2}) + M\delta^{-2}\mathrm{O}(n^{\lambda-\alpha_3}).$$

Choosing $\varepsilon < n^{-s}$ with $s > \lambda - \alpha_2$ we obtain (9).

Case 2. J is an unbounded interval.

According to (8) we have

$$\mu_x(t) = \frac{f(t) - f(x)(t-x) - f''(x)(t-x)^2/2}{(t-x)^2}$$

and applying L'Hospital's rule twice successively it follows again $\lim_{t\to x}\mu_x(t) = 0$. According to (7) we can choose t to be so large that $|f(t)| \leq At^{\beta}$ for some constant $A > 0$ and $0 < x < t/2$. This implies that $|f(t)| \leq 2^{\beta}A\left(\frac{t}{2}\right)^{\beta} < 2^{\beta}A|t-x|^{\beta}$

and hence, from the above definition of μ_x, it results that $\mu_x(t) = \mathrm{O}((t-x)^\gamma)$ as $t \to \infty$ for some $\gamma > 0$. Consequently $|\mu_x(t)| < \varepsilon + A\,|t-x|^\gamma$ and using twice Schwarz inequality we can write successively

$$|(R_n\mu_x)(x)| < \varepsilon\mathrm{O}(n^{-\alpha_2}) + A\sum_{k\in I_n} c_{n,k}p_{n,k}(x)\int_J p_{n,k}(t)|t-x|^{2+\gamma}\mathrm{d}t \le$$

$$\le \varepsilon\mathrm{O}(n^{-\alpha_2}) + A\sum_{k\in I_n} c_{n,k}^2 p_{n,k}(x)\left(\int_J p_{n,k}(t)|t-x|^{2+\gamma}\mathrm{d}t\right)^2 \le$$

$$\le \varepsilon\mathrm{O}(n^{-\alpha_2}) + A\sum_{k\in I_n} c_{n,k}^2 p_{n,k}(x)\left(\int_J p_{n,k}(t)\mathrm{d}t\right)\left(\int_J p_{n,k}(t)(t-x)^{4+2\gamma}\mathrm{d}t\right)$$

$$= \varepsilon\mathrm{O}(n^{-\alpha_2}) + \mathrm{O}(n^{-\alpha_3}).$$

We used both (6) and the definition of $c_{n,k}$. Taking into account that $\lambda < \alpha_3$ and choosing again $\varepsilon < n^{-s}$ with $s > \lambda - \alpha_2$ we arrive at the desired result (9).

This completes the proof of theorem.

3. Applications

We choose $I_n = \{0, 1, \ldots, n\}$, $J = [0, 1]$ and $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. Easily we deduce $c_{n,k} = n+1$, $k = \overline{0, n}$, and D_n becomes M_n the integral analogue of the Bernstein operator studied by Derriennic [5]. The following identities

$$T_{n,1}(x) = \frac{-2x+1}{n+2}, \qquad T_{n,2}(x) = \frac{2nx(1-x) - 6x(1-x) + 2}{(n+2)(n+3)}$$

hold. Taking $\alpha_1 = \alpha_2 = 1$, we obtain $\lambda = 1$, $a(x) = 1 - 2x$, $b(x) = 2x(1-x)$, which imply $\lim\limits_{n\to\infty} n((D_nf)(x) - f(x)) = (1-2x)f'(x) + x(1-x)f''(x)$, that is theorem II.5, [5].

2) The integral analogue of the Baskakov operators was studied by Sahai and Prasad [7] and further on by Sinha, Agrawal and Gupta [2]. Now, we put $I_n = \mathbb{N}$, $J = [0, \infty)$ and $p_{n,k}(x) = \binom{k}{n+k-1}x^k(1-x)^{-n-k}$. Clearly $c_{n,k} = n-1$, $k \in \mathbb{N}$, and after few calculations we obtain

$$T_{n,1}(x) = \frac{2x+1}{n-2},\ n > 2;\ T_{n,2}(x) = \frac{2(n-1)x(1+x) + 2(1+2x)^2}{(n-2)(n-3)},\ n > 3.$$

These lead us to the following results: $\lambda = 1$, $a(x) = 2x+1$, $b(x) = 2x(1+x)$ which imply

$$\lim_{n\to\infty} n((D_nf)(x) - f(x)) = (2x+1)f'(x) + x(1+x)f''(x),$$

that is theorem 1, [7] for $r = 0$.

3) In [8] D.D. Stancu introduced a linear positive operator $L_{m,r}^{\alpha,\beta}$ where r is a non negative integer parameter ($2r < m$) and α, β are real parameters such as $0 \leq \alpha \leq \beta$. This operator can be expressed by means of the fundamental Bernstein polynomials in the form:

$$(L_{m,r}^{\alpha,\beta} f)(x) = \sum_{k=0}^{m-r} b_{m-r,k}(x) \left[(1-x) f\left(\frac{k+\alpha}{m+\beta} \right) + x f\left(\frac{k+r+\alpha}{m+\beta} \right) \right],$$

where $f \in C[0,1]$.

Starting from the above operator, in [4] Chen Wenzhong and Tian Jishan defined a new operator of integral type $P_{n,s}^{*}$ $(s < n/2)$ as follows:

$$(P_{n,s}^{*} f)(x) = \sum_{k=0}^{n-s} \left[(1-x) c_{n,s,k}^{-1} \int_0^1 (1-t) b_{n-s,k}(t) f(t) \mathrm{d}t \; + \right.$$

$$\left. + \; x (c_{n,s,k}^{*})^{-1} \int_0^1 t b_{n-s,k}(t) f(t) \mathrm{d}t \right] b_{n-s,k}(x), \tag{10}$$

where

$$c_{n,s,k} = \frac{n-s-k+1}{(n-s+2)(n-s+1)}, \qquad c_{n,s,k}^{*} = \frac{k+1}{(n-s+2)(n-s+1)}$$

and $b_{n-s,k}(x) = \binom{n-s}{k} x^k (1-x)^{n-s-k}$. The authors computed the degrees of approximations for continuous functions and L_p-functions. In order to apply our theorem we tried to represent (10) under another form and in this way we obtained a surprising result. Firstly we recall two identities:

$$(1-x) b_{n-s,k}(x) = \frac{n-s+1-k}{n-s+1} b_{n-s+1,k}(x),$$

$$x b_{n-s,k}(x) = \frac{k+1}{n-s+1} b_{n-s+1,k+1}(x)$$

We can write successively:

$$(P_{n,s}^{*} f)(x) = \sum_{k=0}^{n-s} \frac{n-s+1-k}{n-s+1} c_{n,s,k}^{-1} b_{n-s+1,k}(x) \; \times$$

$$\times \int_0^1 \frac{n-s+1-k}{n-s+1} b_{n-s+1,k}(t) f(t) \mathrm{d}t \; +$$

$$+ \sum_{k=0}^{n-s} \frac{k+1}{n-s+1} (c_{n,s,k}^{*})^{-1} b_{n-s+1,k+1}(x) \; \times$$

$$\times \int_0^1 \frac{k+1}{n-s+1} b_{n-s+1,k+1}(t) f(t) \mathrm{d}t =$$

$$
\begin{aligned}
&= \frac{n-s+2}{n-s+1}\left\{\sum_{k=0}^{n-s+1}(n-s+1-k)b_{n-s+1,k}(x)\times\right.\\
&\times \int_0^1 b_{n-s+1,k}(t)f(t)\mathrm{d}t+\\
&+\left.\sum_{i=1}^{m+1} ib_{n-s+1,i}(x)\int_0^1 b_{n-s+1,i}(t)f(t)\mathrm{d}t\right\}=(M_{n-s+1}f)(x)-\\
&-\frac{n-s+2}{n-s+1}\sum_{k=0}^{n-s+1} kb_{n-s+1,k}(x)\int_0^1 b_{n-s+1,k}(t)f(t)\mathrm{d}t+\\
&+\frac{n-s+2}{n-s+1}\sum_{i=0}^{n-s+1} ib_{n-s+1,i}(x)\int_0^1 b_{n-s+1,i}(t)f(t)\mathrm{d}t=\\
&= (M_{n-s+1}f)(x).
\end{aligned}
$$

In other words, the operator $P^*_{n,s}$ is right Derriennic's operator M_{n-s+1}, see the first example. What is really amazing about it is the fact that the Chinese authors knew Derriennic's paper (as mentioned in their references) and they obtained many similar results like in [5] but they did not realize the relation $P^*_{n,s} \equiv M_{n-s+1}$.

REFERENCES

1. O. Agratini, *On a generalized Durrmeyer operators*, Bul. St. Univ. Baia Mare, Matematică-Informatică, **12**(1996), 21-30.
2. P.N. Agrawal, V. Gupta and R.P. Sinha, *On simultaneous approximation by modified Baskakov operators*, Bull. Soc. Math. Belg., **43**(1991), 2, 217-231.
3. Wenzhong Chen, *On the integral Meyer-König and Zeller operators*, Approx. Theory Appl., 2(1986), *3*, 7-18.
4. Wenzhong Chen, Tian Jishan, *On approximation properties of Stancu operators of integral type*, Journal of Xiamen University, **26**(1987), *3*, 270-276.
5. M.M. Derriennic, *Sur l'approximation de fonction intégrables sur* [0, 1] *par des polynômes de Bernstein modifies*, Journal Approx. Theory, **31**(1981), 325-343.
6. J.L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: Application à la théorie des moments*, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
7. A. Sahai and G. Prasad, *On simultaneous approximation by modified Lupaș operators*, Journal Approx. Theory, **45**(1985), 122-128.
8. D.D. Stancu, *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, **20**(1983), 2, 211-229.

Received 10 April 1997

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