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Properties Concerning the Baskakov-Beta Operators

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ABSTRACT: Starting from a positive summation integral operator we present linear combinations of these operators which under definite conditions approximate a function more closely then the above operators. Also we establish a connection between the local smoothness of local Lipschitz - α $(0 < \alpha \leq 1)$ functions and the local approximating property.

1 Introduction

V.A. Baskakov [1] has introduced and investigated linear operators of discrete type defined by

$$(V_n f)(x) = (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \quad x \geq 0, \tag{1}$$

for all $n = 1, 2, \ldots$, and for $f \in C_2 := \{f \in C[0,\infty) | \ (1+x^2)^{-1} f(x)$ is convergent as x tends to infinity$\}$. The Banach lattice C_2 is endowed with the norm

$$\|f\|_{C_2} = \sup_{x \geq 0} |f(x)|(1+x^2)^{-1}.$$

In order to obtain an approximation process in the space of integrable functions, A. Sahai and G. Prasad [7] proposed an integral modification of these operators as follows

$$(\widetilde{V}_n f)(x) = (n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt, \quad n = 1, 2, \ldots \tag{2}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}, \quad x \in [0, \infty),$$

and $f \in L_1[0,\infty)$, the space of integrable functions defined on $[0,\infty)$.

By using weight functions of beta-type, the following integral extension was given by V. Gupta [5]

$$(M_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, \quad x \geq 0, \quad n = 1, 2, \ldots, \tag{3}$$

where $b_{n,k}(t) = \dfrac{1}{B(k+1,n)} t^k (1+t)^{-n-k-1}$, $t \geq 0$, and $B(\cdot,\cdot)$ denotes the Beta function. It results

$$\int_0^\infty b_{n,k}(t)dt = 1. \tag{4}$$

These operators are a slight modification of those defined by (2) but some approximation formulas for $M_n f$ are simpler than the corresponding results for $\widetilde{V}_n f$.

We point out that the two modified operators are inspired from the work of Durrmeyer [4] who presented an integral modification of Bernstein polynomials to approximate Lebesgue integrable functions on $[0,1]$. The focus of the present note is on giving combinations of M_n operators which ensure faster convergence in relation to a higher degree of smoothness.

2 Results

Since the classical linear operators like Bernstein, Szasz, Baskakov cannot be used for the investigation of higher orders of smoothness, P.L. Butzer [2] introduced combinations of Bernstein polynomials defined inductively which have higher orders of approximation. Z. Ditzian and V. Totik [3, p.116] extended this method of combinations and defined for the operators L_n, $n \geq 1$, and a fixed integer $r \geq 1$ the combination $L_{n,r}$ as $(L_{n,r}f)(x) = \sum_{i=0}^{r-1} c_i(n)(L_{n_i}f)(x)$, where n_i and $c_i(n)$ satisfy

$$\begin{aligned} &(a)\quad n = n_0 < \cdots < n_{r-1} \leq Kn, \qquad &&(b)\quad \sum_{i=0}^{r-1} |c_i(n)| \leq K, \\ &(c)\quad \sum_{i=0}^{r-1} c_i(n) = 1, &&(d)\quad \sum_{i=0}^{r-1} c_i(n) n_i^{-\rho} = 0 \text{ for } \rho = 1,2,\ldots,r-1, \end{aligned} \tag{5}$$

where K is an absolute constant, $K \in \mathbf{N}$. Also the conditions

$$(c')\ \sum_{i=0}^{r-1} c_i(n) = 1 + \mathbf{o}(n^{-r}), \qquad (d')\ \sum_{i=0}^{r-1} c_i(n) n^{-\rho} = \mathbf{o}(n^{-r}),\ \text{for } \rho = 1,2,\ldots,r-1, \tag{6}$$

can replace (c) and (d) in many cases.

Based on the work of C.P. May [6] we can present a concrete example of a system useful for linear combinations. In this purpose we set $e_j : [0,\infty) \to \mathbf{R}$, $e_j(x) = x^j$, $j \geq 0$, and we fix $k+1$ distinct positive integers namely $d_0, d_1, \ldots, d_k$. We define the numbers $c_i(k)$, $i = \overline{0,k}$ by

$$c_0(0) = 1 \quad \text{and} \quad c_i(k) = d_i^k \prod_{\substack{j=0 \\ j \neq i}}^{k} (d_i - d_j)^{-1}, \quad k \neq 0.$$

These coefficients enjoy the properties

$$\sum_{i=0}^{k} c_i(k) = 1, \quad \sum_{i=0}^{k} c_i(k) d_i^{-\rho} = 0, \text{ for } \rho = 1,2,\ldots,k, \tag{7}$$

in other words the requirements (5-c) and (5-d) are automatically satisfied by our choice ($r := k+1$).

In order to prove this we consider $L_k f$ the Lagrange interpolating polynomial corresponding to the function f and the nodes d_i^{-1}, $i = \overline{0,k}$,

$$(L_k f)(x) = \sum_{i=0}^{k} \frac{w(x)}{(x - d_i^{-1})\frac{dw}{dx}(d_i^{-1})} f(d_i^{-1}),$$

where $w(x) = (x - d_0^{-1})(x - d_1^{-1})\ldots(x - d_k^{-1})$. It is known that for any $\rho \le k$ we have $(L_k e_\rho)(x) = e_\rho(x)$. For $x = 0$ this implies $(L_k e_0)(x) = 1$ and $(L_k e_\rho)(x) = 0$ for $1 \le \rho \le k$. On the other hand we can write

$$(L_k e_\rho)(0) = \sum_{i=0}^{k} \frac{(-1)^k d_i^{k-\rho}}{(d_0 - d_i)\ldots/\ldots(d_k - d_i)} = \sum_{i=0}^{k} c_i(k) d_i^{-\rho},$$

which lead us to the identities from (7).

Further we use the coefficients $c_i(n)$ defined by (5) choosing r a perfect square, $r = s^2$, and replacing the requirement (5-d) with the following

$$\sum_{i=1}^{r-1} \frac{c_i(n) n_i^\rho}{\langle n_i - 1\rangle_m} = 0, \text{ for every } 0 \le \rho \le \left[\frac{m}{2}\right] \text{ and } m = \overline{1, 2\sqrt{r} - 2}, \tag{8}$$

where $\langle\alpha\rangle_m$ represents the lower-factorials defined by $\langle\alpha\rangle_m = \alpha(\alpha-1)\ldots(\alpha-m+1)$ and $[\beta]$ stands for the integral part of β.

Because of $\sum_{m=1}^{2s-2}([m/2]+1) = s^2 - 1$ it results that (8) contains $r-1$ relations.

Let φ be the function defined on $[0,\infty)$ by $\varphi(x) = \sqrt{x(x+1)}$, $x \ge 0$. Actually $V_n((e_1 - xe_0)^2; x) = \varphi^2(x)/n$ and φ becomes the step weight function of the Baskakov operators and it controls their rate of convergence. For a fix $r = s^2$ we define a linear combination of Baskakov-Beta operators as follows

$$(M_{n,r} f)(x) = \sum_{i=0}^{r-1} c_i(n)(M_{n_i} f)(x), \tag{9}$$

where n_i and $c_i(n)$ satisfy (5-a,b,c) and (8). It is clear that for $r = 1$ one obtains $M_{n,0} = M_n$. Further we consider $s > 1$. Since $M_n e_0 = e_0$ for every natural n, the relations (5-c) and (9) imply

$$M_{n,r} e_0 = e_0. \tag{10}$$

Lemma 1: ([5]) *Let the m^{th} order moment for the operator M_n be defined by*

$$T_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t)(t-x)^m dt.$$

Then $T_{n,0}(x) = 1$, $T_{n,1}(x) = \dfrac{1+x}{n-1}$, $(n > 1)$, $T_{n,2}(x) = \dfrac{2(n+1)x^2 + 2(n+2)x + 2}{(n-1)(n-2)}$, $(n > 2)$, and for $n > m+1$ there holds the recurrence relation

$$(n - m - 1)T_{n,m+1}(x) = x(x+1)\left(\frac{d}{dx}T_{n,m}(x) + 2mT_{n,m-1}(x)\right) + ((m+1)(2x+1) - x)T_{n,m}(x).$$

Remarks. (i) $T_{n,m}(x)$ is a polynomial in x of m degree whose coefficients depend on n but are bounded for all n.

(ii) $(n-1)(n-2)\dots(n-m)T_{n,m}(x)$ is a polynomial in n of degree less or equal to $[m/2]$. Consequently for each $x \geq 0$, $T_{n,m}(x) = \mathcal{O}(n^{[m/2]-m}) = \mathcal{O}(n^{-[(m+1)/2]})$.

The above remarks together with (8) guarantee

$$M_{n,r}((\cdot - x)^k; x) = \sum_{i=0}^{r-1} c_i(n) T_{n_i,k}(x) = 0, \quad \text{for every} \quad k = \overline{1, 2s-2}. \tag{11}$$

Theorem 1. *Let $M_{n,r}$ be defined by (9) and let f be bounded and integrable on $[0,\infty)$. If f has a derivative of $(2s-2)$ order at a point $x \geq 0$ then*

$$|(M_{n,r}f)(x) - f(x)| = \mathcal{O}(n^{-s+1}).$$

Proof. At first we use the Taylor's expansion of f

$$f(t) = \sum_{i=0}^{2s-2} \frac{(t-x)^i}{i!} f^{(i)}(x) + \theta_x(t)(t-x)^{2s-2},$$

where $\theta_x(t) \to 0$ as $t \to x$ and it is a bounded function. Applying the linear operator $M_{n,r}$ we obtain

$$(M_{n,r}f)(x) - f(x) = f(x)((M_{n,r}e_0)(x) - 1) +$$

$$+ \sum_{i=1}^{2s-2} \frac{1}{i!} f^{(i)}(x) M_{n,r}((\cdot - x)^i; x) + M_{n,r}((\cdot - x)^{2s-2}\theta_x; x)$$

and taking into account both (10), (11) and (9) we have

$$|(M_{n,r}f)(x) - f(x)| \leq \sum_{i=0}^{r-1} |c_i(n)| M_{n_i}((\cdot - x)^{2s-2}|\theta_x|; x).$$

From (5-a,b) and Remarks (ii) the result follows. □

This result indicates that $M_{n,r}$ comparatively to M_n improves the rate of convergence for smooth functions.

Now we return at M_n operators to present a new property of them.

Lemma 2. *If M_n is defined by (3) then for every $0 < \alpha \leq 1$ and $h \geq 0$ one has*

$$M_n(h^\alpha; x) \leq (M_n(h^2; x))^{\alpha/2}.$$

Proof. Considering $r := 2/\alpha$ in the relation $1/r + 1/s = 1$, $r > 0$, $s > 0$, which characterizes Hölder's inequality, from (4) we get

$$\int_0^\infty h^\alpha(t) b_{n,k}(t)dt \leq \left(\int_0^\infty h^2(t) b_{n,k}(t)dt\right)^{\alpha/2}.$$

By using this inequality as well as Hölder's and knowing that $\sum_{k=0}^{\infty} p_{n,k}(x) = 1$ we get

$$M_n(h^\alpha; x) \le \sum_{k=0}^{\infty} p_{n,k}^{\alpha/2+1/s}(x) \left(\int_0^\infty h^2(t) b_{n,k}(t) dt \right)^{\alpha/2} \le (M_n(h^2; x))^{\alpha/2}.$$

The proof is complete. □

As a consequence of Lemma 2 we obtain

$$M_n(|e_1 - xe_0|^\alpha; x) \le T_{n,2}^{\alpha/2}(x), \quad n = 3, 4, \ldots, \quad x \ge 0. \tag{12}$$

For our further purpose we need the following definition.

A continuous function f defined on J is locally $Lip\alpha$ on E $(0 < \alpha \le 1,\ E \subset J)$ if it satisfies the condition

$$|f(x) - f(y)| \le M_f |x - y|^\alpha, \ (\forall)\ (x, y) \in J \times E \tag{13}$$

where M_f is a constant depending only on α and f.

Theorem 2. *Let M_n, $n \ge 4$, be given by (3), $0 < \alpha \le 1$ and E be any subset of $J = [0, \infty)$. If f is locally $Lip\alpha$ on E then we have*

$$|(M_n f)(x) - f(x)| \le \left(\frac{\sqrt{2}}{n-2} \right)^\alpha M_f \{1 + (2(n-2)\varphi^2(x))^{\alpha/2}\} + 2M_f (d(x, E))^\alpha,$$

where $d(x, E)$ is the distance between x and E defined as $d(x, E) = \inf\{|x - y| :\ y \in E\}$.

Proof. By using the continuity of f it is obvious that (13) holds for any $x \in [0, \infty)$ and $y \in \overline{E}$, the closure of the set E. Let $(x, x_0) \in [0, \infty) \times \overline{E}$ be so that $|x - x_0| = d(x, E)$. On the other hand we can write $|f - f(x)| \le |f - f(x_0)| + |f(x_0) - f(x)| e_0$ and applying the linear and positive operator M_n we have

$$|(M_n f)(x) - f(x)| \le M_n(|f - f(x_0)|; x) + |f(x) - f(x_0)| \le$$

$$\le M_n(M_f |e_1 - x_0 e_0|^\alpha; x) + M_f |x - x_0|^\alpha. \tag{14}$$

At this point we use the classical inequality

$$(a + b)^\alpha \le a^\alpha + b^\alpha, \quad a \ge 0, \quad b \ge 0, \quad 0 < \alpha \le 1,$$

which implies $|t - x_0|^\alpha \le |t - x|^\alpha + |x - x_0|^\alpha$, $t \ge 0$, and further

$$M_n(|e_1 - x_0 e_0|^\alpha; x) \le M_n(|e_1 - xe_0|^\alpha; x) + |x - x_0|^\alpha \le T_{n,2}^{\alpha/2}(x) + |x - x_0|^\alpha.$$

The last increase is based on (12). The expression of $T_{n,2}$ guarantees

$$T_{n,2}(x) \le \frac{4\varphi^2(x)}{n-2} + \frac{2}{(n-2)^2}, \quad n \ge 4.$$

Gathering the above relations, returning at (14) and using again (15) we obtain the desired result. □

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