DEMONSTRATIO MATHEMATICA
Vol. XXXII No 4 1999

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NOTE ON A CLASS OF OPERATORS ON INFINITE INTERVAL

1. Introduction

A. Lupaş and M. Müller introduced [5] an integral operator defined by

$$(1)\qquad (G_n f)(x) = \int_0^\infty g_n(x,u) f\left(\frac{n+1}{u}\right) du, \quad n = 1, 2, \dots,$$

where

$$g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n$$

and f is bounded for $x > 0$ and locally integrable on the half-line $x > 0$. We mention that this Gamma operator differs substantially from the "Gamma" operator which has already been considered by Feller [4].

Based on (1), S.M. Mazhar [6] defined and studied a new linear operator which reproduces linear functions, namely:

$$(2)\qquad (F_n)(x) = \int_0^\infty \int_0^\infty g_n(x,u) g_{n-1}(u,t) f(t) du dt =$$

$$= \frac{(2n)!}{n!(n-1)!} \int_0^\infty \frac{w^{n-1}}{(1+w)^{2n+1}} f(wx) dw, \quad x > 0,\ n > 1,$$

for any f for which the integral is convergent.

In this paper, by using a probabilistic method, we construct a family of linear and positive operators which includes (2). Also, we point out some new properties of the F_n operators such as the preservation of the monotonicity and convexity and we establish a connection between the local smoothness of function and the local approximation property.

1991 *Mathematics Subject Classification*: 41A35.

2. Probabilistic background

In what follows we denote by $\Gamma(\cdot)$ and $B(\cdot,\cdot)$ Euler's functions, gamma respectively beta. Let X_a, $a>0$, be a random variable so that it has the gamma distribution with density g_a, i.e. $g_a(x)=(\Gamma(a))^{-1}x^{a-1}e^{-x}$ for $x>0$ and $g_a(x)=0$ otherwise. Further on, we present a family of one-dimensional operators by the formula

$$(L_{a,b}f)(x)=E[f(X_a/X_b)],$$

where f is any real measurable function defined on $(0,\infty)$ such that $E[f(X_a/X_b)]<\infty$, where E stands for the mathematical expectation.

From the properties of the operator E it is obvious that $L_{a,b}$ is linear and positive and consequently becomes monotone.

For any independent variables X_a, X_b, $a>0$ and $b>0$, the random variable X_a/X_b has the beta distribution $g_{(a,b)}$. Indeed, for $y>0$ we have

$$(3) \qquad g_{(a,b)}(y)=\int_{\mathbf{R}}|x|g_a(xy)g_b(x)dx=$$

$$=\frac{y^{a-1}}{\Gamma(a)\Gamma(b)}\int_0^\infty x^{a+b-1}e^{-(y+1)x}dx=\frac{1}{B(a,b)}\frac{y^{a-1}}{(1+y)^{a+b}},$$

and $g_{(a,b)}(y)=0$ otherwise.

We will indicate some particular cases.

A. Choosing $a=mx$ and $b=m+1$ with $x>0$ and $m=1,2,3,\ldots$ we obtain

$$(L_m f)(x)=\frac{1}{B(mx,m+1)}\int_0^\infty\frac{t^{mx-1}}{(1+t)^{mx+m+1}}f(t)dt,$$

where f belongs to $B(0,\infty)$, the linear space of all real measurable bounded functions defined on $(0,\infty)$. These operators were investigated in 1995—see [7].

B. Choosing $a=x/\alpha$ and $b=1/\alpha$ with $x>0$ and $\alpha>0$ we get

$$(L_{x/\alpha,1/\alpha}f)(x)\equiv(T^\alpha f)(x),\quad f\in B(0,\infty).$$

This class of Bernstein-type operators was introduced in 1993 and studied by J.A. Adell, F.G. Badia, Jesús de la Call and M. San Miguel (see [1], [2], [3]). If f is defined in $x=0$ as well, we set $(T^\alpha f)(0)=f(0)$.

C. Firstly, for any $x>0$ and any real function acting on the half line $(0,\infty)$ we define φ_x^f as follows $\varphi_x^f(u)=f(xu)$, $u>0$.

Now, we choose $a=n$, $b=n+1$, $n=1,2,3,\ldots$, substitute f by φ_x^f and taking into account (3) we get

$$E[\varphi_x^f(X_n/X_{n+1})]=\frac{1}{B(n,n+1)}\int_0^\infty\frac{y^{n-1}}{(1+y)^{2n+1}}f(xy)dy.$$

Using the relation $B(n, n+1) = n!(n-1)!/(2n)!$, we can write (2) as follows

$$(L_{n,n+1}f)(x) = (F_n f)(x), \quad x > 0.$$

We notice that if f is defined in $x = 0$ as well, we can take $(L_n f)(0) = f(0)$. Regarding F_n, $n > 1$, we recall some usefull formulas [6]

$$F_n(e_0, x) = 1, \quad F_n(e_1, x) = x, \quad F_n(e_2, x) = \frac{n+1}{n-1}x^2, \tag{4}$$

where $e_j(x) = x^j$, $j = 0, 1, 2$, are the so called test-functions.

3. Results concerning the F_n operators

First we denote by $\mu_{n,m}(x)$ the m^{th} central moment of F_n operators, i.e. $\mu_{n,m}(x) = F_n((e_1 - x)^m, x)$. It was already established that $\mu_{n,2}(x) = 2x^2/(n-1)$. Further on we indicate the values of the central moments of all orders.

THEOREM 1. *The m^{th} central moment of the operators defined by* (2) *has the following value*

$$\mu_{n,m}(x) = x^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{(n)_i}{\langle n \rangle_i}, \ n > 1, \ m > 1 \textit{ integers and } x \in (0, \infty),$$

where $(n)_i$ and $\langle n \rangle_i$ represent the upper-factorials and respectively the lower-factorials.

Proof. Taking into account both the identity $B(p,q) = \int_0^\infty t^{p-1}(1-t)^{-p-q}dt$ and the relation (2) we have

$$\mu_{n,m}(x) = \frac{x^m}{B(n, n+1)} \int_0^\infty \frac{w^{n-1}}{(1+w)^{2n+1}}(w-1)^m dw =$$
$$= x^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{B(n+i, n+1-i)}{B(n, n+1)}$$

which lead us to the desired results. □

Concerning the remainder of the approximation formula

$$f(x) = (F_n f)(x) + (R_n f)(x), \tag{5}$$

which has the degree of exactness $N = 1$, we can give an integral representation.

THEOREM 2. *If the function f has a continuous second derivative then the remainder of the formula* (5) *has the following integral form*

$$(R_n f)(x) = \int_0^\infty K_n(t, x) f''(t) dt,$$

where $K_n(t,x) = (R_n\theta_x)(t)$ *and* $\theta_x(t) = (x-t)_+ = (x-t+|x-t|)/2$. *Also it results the identity*

$$\int_0^\infty K_n(t,x)dt = -\frac{x^2}{n-1}.$$

Proof. The first relation follows directly from the well-known theorem of Peano.

Since for any fixed point $x > 0$ we have $K_n(t,x) \le 0$, then the mean value theorem of the integral calculus and (5) implies $f(x) = (F_nf)(x) + f''(\xi)\int_0^\infty K_n(t,x)dt$, where $\xi \in (0,\infty)$. Choose $f = e_2$. The above identity with (4) allows us to obtain the integral of Peano's kernel that is the second identity of our theorem. □

In the following, we establish two new properties of the F_n operators.

THEOREM 3. *Let* $(F_n)_{n>1}$ *be defined by (2) and* f *be a function defined on* $(0,\infty)$ *such as* $F_n(|f|,x) < \infty$ *for all* $x > 0$.

(*i*) *If* f *is an increasing function then every* F_nf *is increasing as well,* $n = 1,2,\ldots$

(*ii*) *If* f *is a convex function then every* F_nf *is convex as well,* $n = 1,2,\ldots$

Proof. If f is increasing then φ_x^f is also increasing and the first statement of the theorem can be easily checked. Therefore we omit it. In order to prove the second statement we observe that the operator (2) is given by

$$F_n(f,x) = E[f(xY_n)], \quad x > 0,$$

where Y_n is the random variable with density $g_{(n,n+1)}$ given by (3). Let f be convex, let $0 \le \alpha \le 1$ and $x_1 \ne x_2$. Then

$$f((\alpha x_1 + (1-\alpha)x_2)Y_n) \le \alpha f(x_1Y_n) + (1-\alpha)f(x_2Y_n),$$

and taking the expectation one obtains

$$F_n((\alpha x_1 + (1-\alpha)x_2)Y_n) \le \alpha F_n(f,x_1) + (1-\alpha)F_n(f,x_2),$$

and the convexity of F_nf is established. □

From the above theorem it follows directly that:

(i) if f is decreasing then F_nf are decreasing;

(ii) if f is concave then F_nf are concave.

The previous results about F_n were concerned the global smoothness of functions and the global approximation property. Now we are going to discuss a connection between local smoothness of functions and local convergence of F_n. For this purpose we recall that a continuous function f is locally $Lip\alpha$ $(0 < \alpha \leq 1)$ on $E \subset (0, \infty)$ if it satisfies the condition

$$|f(x) - f(y)| \leq M_f |x - y|^\alpha, \quad x > 0,\ y \in E, \tag{6}$$

where M_f is a constant depending only on α and f.

THEOREM 4. *Let F_n be given by (2), $0 < \alpha \leq 1$ and E be a subset so that $\overline{E} \subset (0, \infty)$ ($\overline{E}$ stands for the closure). If $f \in C(0, \infty)$ is locally $Lip\alpha$ on E and $F_n(|f|, x) < \infty$ for all $x > 0$ then we have*

$$|(F_n f)(x) - f(x)| \leq M_f \left(\left(\frac{2x^2}{n-1} \right)^{\alpha/2} + 2d^\alpha(x, E) \right),$$

where $d(x, E) := \inf\{|x - y| :\ y \in E\}$.

Proof. If f is continuous then (6) holds for any $x > 0$ and $y \in \overline{E}$. Fix $x > 0$ and let $x_0 \in \overline{E}$ be such that $|x - x_0| = d(x, E)$. Using the properties of the linear positive operator F_n we deduce that

$$\begin{aligned} |(F_n f)(x) - f(x)| &\leq F_n(|f - f(x_0)|, x) + |f(x) - f(x_0)| \leq \\ &\leq M_f \{F_n(|e_1 - x_0|^\alpha, x) + |x - x_0|^\alpha\}. \end{aligned} \tag{7}$$

The well-known inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$, $\alpha \in (0, 1]$, $a \geq 0$, $b \geq 0$, implies $|t - x_0|^\alpha \leq |t - x|^\alpha + |x - x_0|^\alpha$, $t > 0$, and further $F_n(|e_1 - x_0|^\alpha, x) \leq \mu_{n,2}^{\alpha/2}(x) + |x - x_0|^\alpha = (2x^2/(n-1))^\alpha + |x - x_0|^\alpha$. We have used the Hölder's inequality ($r := 2/\alpha$, $s > 0$, $1/r + 1/s = 1$) and the fact that $(F_n e_0)(x) = 1$. Returning to (7) one can check that the conclusion follows. □

In particular, if we assume that f is defined in $x = 0$ then, setting $(F_n f)(0) = f(0)$ and choosing $E = [0, \infty)$, we can infer:

If $f \in C[0, \infty)$, $F_n(|f|, x) < \infty$, $x \geq 0$, and $\omega_1(f, t) = O(t^\alpha)$, $\alpha \in (0, 1]$, then there exists a constant M_f independent of n and x so that $|(F_n f)(x) - f(x)| \leq M_f (2x^2/(n-1))^{\alpha/2}$, $x \geq 0$. Here $\omega_1(f, t) = \sup_{0 < h \leq t}\{|f(x + h) - f(x)| :\ x \geq 0\}$.

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Received December 23, 1998.