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APPROXIMATION PROPERTIES OF A CLASS OF LINEAR OPERATORS

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Abstract. In this paper we deal with a class of linear operators of integral type. We evaluate the order of approximation and indicate conditions which ensure the uniform convergence of the sequence. Also, we apply our result to operators which represent a generalization of Stancu's operators.

Introduction

Durrmeyer [1] defined a new kind of modified Bernstein polynomial operators on $L_1[0,1]$, the space of Lebesgue integrable functions on [0,1], as:

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1\binom{n}{k}t^k(1-t)^{n-k}f(t)dt. \qquad (1)$$

The aim of this note is to present a general class of linear positive operators $(L_n)_{n\geq 1}$ of integral type. This construction contains as particular cases well-known operators introduced and studied during the time by many authors. We evaluate the order of approximation in terms of the moduli of smoothness ω, ω_2, and indicate sufficient conditions which ensure the uniform convergence of the sequence. In the last section of this paper we apply our result to operators which represent a generalization of Stancu's operators. We mention that our estimation improves a previous result.

1. Examples

Let J and D be subintervals of real axis and for each integer $n\geq 1$ let I_n be a set of indexes. We consider $p_{n,k}, q_{n,k}$ real non-negative valued functions with the properties $p_{n,k}\in \mathbb{C}(J)$, $q_{n,k}\in \mathbb{C}(J\times D)$, and for any $x\in J$, $q_{n,k}(x,\cdot)$ is bounded on D, where $n\geq 1$ and $k\in I_n$. The operators in question are defined on the space

$$L_1(D) = \{f : D\to \mathbb{R} |\int_D |f(x)|dx < \infty\}$$

by

$$(L_n f)(x) = \sum_{k \in I_n} p_{n,k}(x) \int_D q_{n,k}(x,y) f(y) dy. \tag{2}$$

It is obvious that this operator is linear and positive. Now we shall analyse three particular cases.

A. Firstly, we choose $D = J = [0,1]$ and $I_n = \{0, 1, 2, \ldots, n\}$. If we put

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{and} \quad q_{n,k}(x,y) = (n+1) p_{n,k}(y)$$

we obtain the operator defined by (1). Derriennic [2] studied these modified Bernstein polynomials. He demonstrated that if the derivative $d^r f / dx^r$ with $r \geq 0$ is continuous on [0,1], then $(d^r/dx^r) M_n f$ converges uniformly on [0,1] and

$$\sup_{x \in [0,1]} |(M_n f)(x) - f(x)| \leq 2\omega(n^{-1/2})$$

where ω is the modulus of continuity of f. He has also proved that if f belongs to Sobolev space $W^{l,p}(0,1)$ with $l \geq 0$, $p \geq 1$, then $M_n f$ converges to f in $W^{l,p}(0,1)$.

B. Secondly, we choose $D = J = [0, \infty)$ and $I_n = \mathbb{N}$. Now we take

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad q_{n,k}(x,y) = (n-1) p_{n,k}(y)$$

then L_n become operators proposed by Sahai and Prasad [3] and termed as modified Lupaş operators. The authors established theorems concerning asymptotic approximation and error estimation in the simultaneous approximation by these operators. Later, Sinha, Agrawal, Sahai and Gupta [4, 5] enlarged the study upon L_n. Using the device of Steklov means they obtained the estimate of error in L_p-approximation in terms of higher order integral modulus of smoothness. We note that the above operators also should be called modified Baskakov operators because Baskakov [6] has introduced the operators:

$$(K_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

If we replace $f\left(\frac{k}{n}\right)$ by $(n-1)\int_0^\infty p_{n,k}(t) f(t) dt$, we obtain the operators discussed at this example.

C. Finally, we choose $D = J = [0,1]$ and $I_n = \{0, 1, 2, \ldots, n-r\}$ where r is an integer parameter such that $2r < n$. We define:

$$p_{n,k}(x) = \binom{n-r}{k} x^k (1-x)^{n-r-k}$$

and

$$q_{n,k}(x,y) = (1-x)A_{n,r,k}(1-y)p_{n,k}(y) + xB_{n,r,k}yp_{n,k}(y) \tag{3}$$

where the constants $A_{n,r,k}$, $B_{n,r,k}$ have the following form:

$$A_{n,r,k} = \frac{(n-r+1)(n-r+2)}{n-r-k+1}, \quad B_{n,r,k} = \frac{(n-r+1)(n-r+2)}{k+1}, \quad 0 \le k \le n-r.$$

This operator was studied by Chen Wenzhong and Tian Jishan [7]. They computed the degrees of approximations for continuous functions and L_p-functions. In this case L_n represents a generalization of an operator introduced by Stancu [8]. The expression of Stancu's operator is the following:

$$(L_{m,r}^{\alpha,\beta} f)(x) = \sum_{k=0}^{m-r} \binom{m-r}{k} x^k (1-x)^{m-r-k} \times$$
$$\times \left[(1-x) f\left(\frac{k+\alpha}{m+\beta}\right) + x f\left(\frac{k+r+\alpha}{m+\beta}\right)\right]$$

where r is a non-negative integer $(2r < n)$ and α, β are real parameters so that $0 \le \alpha \le \beta$.

$L_{m,r}^{\alpha,\beta}$ was introduced using a probabilistic method. A special attention was given to the case of the operator $L_{m,r} = L_{m,r}^{0,0}$. Stancu proved that the remainder of the approximation formula of a function $f \in \mathbb{C}[0,1]$ by $L_{m,r}f$ can be represented either by means of divided differences, or by an integral form, obtained by using a classical theorem of Peano. He also showed that $L_{m,r}$ admits the variation diminishing property, he determined the point spectrum of $L_{m,r}$ and constructed a quadrature formula.

2. The Results

In this section we are concerned with the estimate of the order of approximation of a function f by means of the linear operator L_n.

Theorem A. *Let be the sequence defined by (2) and $K \subset J$ be a compact with the property that f is continuous on K. For every $x \in K$ let*

$$\int_D q_{n,k}(x,y)(x-y)^r dy = \mu_{n,k}(r;x), \quad r \in \{0,2\} \tag{4}$$

and

$$\sum_{k \in I_n} p_{n,k}(x)\mu_{n,k}(0;x) = a_n(x). \tag{5}$$

If

i) two real numbers A, B exist so that $0 < A \le a_n(x) \le B$, $x \in K$,

ii) $\alpha > 0$ exists so that $\sum_{k \in I_n} p_{n,k}(x)\mu_{n,k}(0;x)\mu_{n,k}(2;x) = 0(n^{-\alpha})$,

then the following inequality

$$|(L_n f)(x) - f(x)| \le B\gamma_n |f(x)| + (B + 0(n^{-\alpha+1/2}))\omega(f, n^{-1/2}) \tag{6}$$

holds, where

$$\gamma_n = \sup_{x\in K} |1 - a_n^{-1}(x)|.$$

Proof. By using (5) we get:

$$f(x) = \frac{f(x)}{a_n(x)} \sum_{k\in I_n} p_{n,k}(x) \int_D q_{n,k}(x,y)dy.$$

Taking into account the above relation and (2) we deduce:

$$|(L_n f)(x) - f(x)| \le \sum_{k\in I_n} p_{n,k}(x) \int_D q_{n,k}(x,y) \left| f(y) - \frac{f(x)}{a_n(x)} \right| dy. \tag{7}$$

We shall use the modulus of continuity ω defined by:

$$\omega(\delta) = \omega(f,\delta) = \sup |f(x'') - f(x')|,$$

where x' and x'' are points from K so that $|x'' - x'| < \delta$, δ being a positive number. Using the following well-known properties of ω

$$|f(x'') - f(x')| \le \omega(|x'' - x'|) \le (1 + \delta^{-1}|x'' - x'|)\omega(f,\delta)$$

and the definition of γ_n we can write successively:

$$\left| f(y) - \frac{f(x)}{a_n(x)} \right| \le |f(y) - f(x)| + |f(x)| \left| 1 - \frac{1}{a_n(x)} \right| \le$$
$$\le (1 + \delta^{-1}|x - y|)\omega(f,\delta) + \gamma_n |f(x)|. \tag{8}$$

Substituting (8) in relation (7) we obtain:

$$|(L_n f)(x) - f(x)| \le \gamma_n |f(x)| a_n(x) + (a_n(x) +$$
$$+\delta^{-1} \sum_{k\in I_n} p_{n,k}(x) \int_D q_{n,k}(x,y)|x-y|dy)\omega(f,\delta). \tag{9}$$

Now, we apply the Cauchy inequality:

$$\int_D q_{n,k}(x,y)|x-y|dy \le \int_D q_{n,k}(x,y)dy \int_D q_{n,k}(x,y)(x-y)^2 dy =$$
$$= \mu_{n,k}(0;x)\mu_{n,k}(2;x). \tag{10}$$

If we insert $\delta = n^{-1/2}$, the relations (9), (10) and the hypotheses of this theorem lead us to the desired result.

The modulus of smoothness ω_2 is also frequently used in quantitative approximation. Gavrea and Raşa [9] established the inequality:

$$\omega(f,\delta) \le \left(3 + \frac{I(K)}{\delta}\right)\omega_2(f,\delta) + \frac{6\delta}{I(K)}\|f\|, \quad 0 < \delta \le I(K)$$

where $I(K)$ represents the length of the interval compact K and $\|\cdot\|$ is sup-norm on $\mathbb{C}(K)$.

We mention that:

$$\omega_2(f,\delta) = \sup\{|f(x+2t) - 2f(x+t) + f(x)| : \ 0 \le t \le \delta, \ x \in K, \ x + 2t \in K\}.$$

After a few computations we can state:

Theorem B. *If the notations and conditions required by theorem A work, the following inequality*

$$|(L_nf)(x) - f(x)| \leq B(\gamma_n + 6I^{-1}(K)n^{-1/2} + O(n^{-\alpha}))\|f\|+$$

$$+B(3 + I(K)n^{1/2} + O(n^{-\alpha+1}))\omega_2(f, n^{-1/2})$$

holds.

Once inequality (6) is known, we can easily obtain:

Corollary. *Under the conditions of theorem A, if $\alpha > 1/2$ and γ_n converges uniformly to zero, then*

$$\lim_{n\to\infty} (L_nf)(x) = f(x),$$

the convergence being uniform on K.

3. Application

We come back at example C. It is known that the Beta function is defined as

$$B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$$

and $\Gamma(s) = (s-1)!$ for $s \geq 1$ integer.

For $q_{n,k}$ defined in (3), this formula helps us to prove that

$$\int_0^1 q_{n,k}(x,y)dy = 1 = \mu_{n,k}(0;x),$$

$$\int_0^1 yq_{n,k}(x,y)dy = \frac{x+k+1}{n-r+3},$$

$$\int_0^1 y^2q_{n,k}(x,y)dy = \frac{2(k+2)x + (k+1)(k+2)}{(n-r+3)(n-r+4)}.$$

Consequently, we get:

$$\mu_{n,k}(2;x) = \frac{n-r+1}{n-r+3}x^2 - 2\frac{(n-r+3)k + (n-r+2)}{(n-r+3)(n-r+4)}x + \frac{(k+1)(k+2)}{(n-r+3)(n-r+4)}.$$

In this case we have $a_n(x) = 1, \; A = B = 1, \; \gamma_n = 0$.

By using the identities

$$\sum_{k=1}^{n-r} k\binom{n-r}{k}x^k(1-x)^{n-r-k} = (n-r)x,$$

$$\sum_{k=2}^{n-r} k(k-1)\binom{n-r}{k}x^k(1-x)^{n-r-k} = (n-r)(n-r-1)x^2,$$

we obtain

$$\sum_{k=0}^{n-r} p_{n,k}(x)\mu_{n,k}(0;x)\mu_{n,k}(2;x) =$$

$$= \frac{2(n-r-2)}{(n-r+3)(n-r+4)}\left(x - x^2 + \frac{1}{n-r-2}\right) \leq \frac{1}{2(n-r+5)}$$

because $x - x^2 \leq 1/4,\ x \in [0,1]$.

According to theorem A, all these relations lead us to the following inequality:

$$|(L_n f)(x) - f(x)| \leq \left(1 + \frac{\sqrt{n}}{2(n-r+5)}\right)\omega(f, n^{-1/2}).$$

Our result improves the one given in [7] because there the coefficient of ω was found as being 2.

We mention that in the paper [10] an extension of operator $L_{m,r}^{\alpha,\beta}$ in the sense of Kantorovich was presented.

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