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SMOOTHNESS PROPERTIES OF POSITIVE SUMMATION INTEGRAL OPERATORS

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In this paper we are dealing with approximation by summation integral operators. We show the connections between the local smoothness of the approximated function and the rate of its local approximation. The direct theorem is obtained in a general case. Also, an inverse result is presented under certain conditions imposed on the sequence of operators, the most important being the commutativity of the operators and a restriction on the second order moments of the operators.

1. Introduction

Let J be a given interval on the real line. To approximate continuous functions f on J, we use a sequence $(l_n)_{n\geq 1}$ of linear positive operators of discrete type, that is, operators of the form

$$(1) \qquad (l_n f)(x) = \sum_{k\in I_n} u_{n,k}(x) f(x_{n,k}),$$

where $I_n \subset \mathbb{N}$ is a set of indices, $u_{n,k}$ are non-negative functions in the space $C(J)$ and the knots $x_{n,k}$ are suitably chosen in J. For a given compact K in J we study the approximation of functions from $C(K)$ with respect to the uniform norm $\|\cdot\|_K$. To determine a positive approximation process on $C(K)$, the above sequence has to satisfy the Korovkin conditions $\|l_n e_i - e_i\|_K \to 0$,

$i = 0, 1, 2$, where $e_i(t) = t^i$, $t \in K$. Further on, we assume that l_n reproduces every constant function. In other words,

$$\sum_{k \in I_n} u_{n,k}(x) = 1, \quad x \in J. \tag{2}$$

In order to generalize l_n to a summation-integral operator L_n, we follow Durrmeyer and use a non-negative family $\{\omega_{n,k}\}$ of functions from Lebesgue space $L_1(J)$ and normalized by

$$\int_J \omega_{n,k}(t)\, dt = 1, \quad k \in I_n. \tag{3}$$

Then we define L_n as

$$(L_n f)(x) = \sum_{k \in I_n} u_{n,k}(x) \int_J \omega_{n,k}(t) f(t)\, dt. \tag{4}$$

In [1] we indicated conditions that ensure the convergence of the sequence (4). In [2] the authors have presented the asymptotic properties of L_n in the case $I_n = \{0, 1, \ldots, n\}$ and $J = [0, 1]$. The general class of operators we are studying includes those considered in the literature under the name of "modified operators" being the integral analogue of the Bernstein, Baskakov, Meyer – König and Zeller, Szasz operators (see, respectively, [3], [7], [8], [6]). The aim of the present paper is to give an equivalence between the local smoothness of functions and the local convergence of L_n operators. For this purpose, we need the following definition.

Definition. A continuous function f defined on J is *locally* Lipα *on* E $(0 < \alpha \leq 1,\ E \subset J)$ if it satisfies the condition

$$|f(x) - f(y)| \leq M_f |x - y|^\alpha, \quad \forall\ (x, y) \in J \times E, \tag{5}$$

where M_f is a constant depending only on α and f.

2. A direct theorem

The relations (2) and (3) guarantee $(L_n e_0)(x) = 1$. Further we shall need the second central moment of L_n defined by $\mu_{n,2}(x) := L_n((e_1 - x e_0)^2, x)$. Since we want to show that the sequence $(L_n)_{n \geq 1}$ converges to the identity operator, it is necessary to assume that a continuous function φ exists so that $\lim_{n\to\infty} n\mu_{n,2}(x) = \varphi(x)$. Actually, we can represent $n\mu_{n,2}$ in the form

$$n\mu_{n,2}(x) = \varphi(x) + \sum_{i=1}^{l} \frac{\varphi_i(x)}{n^i}, \quad x \in J, \tag{6}$$

where $\varphi_i \in C(J)$, $i = \overline{1,l}$. It is clear that $\varphi \geq 0$ and we will call $\varphi^{1/2}$ the step-weight function related to L_n operators.

First, we mention an immediate consequence from Hölder's inequality.

Lemma 1. *If L_n is defined by (4), then for every $0 < \alpha \leq 1$ we have*

$$L_n(h^\alpha, x) \leq ((L_n h^2)(x))^{\frac{\alpha}{2}}$$

where $h \geq 0$ and $\{h^2 \omega_{n,k}\}_{k\in I_n} \subset L_p(J)$ ($p = 1$ or $p = \infty$, if J is a bounded, respectively, unbounded interval).

Indeed, an application of Hölder's inequality with parameters $r = 2/\alpha$ and $s > 0$ $(1/r + 1/s = 1)$ gives on the basis of (3)

$$\int_J h^\alpha(t)\omega_{n,k}(t)\,dt \leq \left(\int_J h^2(t)\omega_{n,k}(t)\,dt\right)^{\frac{\alpha}{2}}.$$

By using this relation and Hölder's inequality, from (2) we get

$$L_n(h^\alpha, x) \leq \sum_{k\in I_n} u_{n,k}^{\frac{\alpha}{2}+\frac{1}{s}}(x) \left(\int_J h^2(t)\omega_{n,k}(t)\,dt\right)^{\frac{\alpha}{2}} \leq (L_n(h^2, x))^{\frac{\alpha}{2}},$$

which was to be shown.

As a consequence of Lemma 1 we obtain

$$L_n(|e_1 - x|^\alpha, x) \leq \mu_{n,2}^{\frac{\alpha}{2}}(x), \quad x \in J. \tag{7}$$

Next we give an estimate of the rate of approximation in terms of $\mu_{n,2}$. Denote by $d(x, E)$ the distance between x and E, that is,

$$d(x, E) = \inf\{|x - y| : y \in E\}.$$

Theorem 1. *Let L_n be given by (4), $0 < \alpha \leq 1$ and E be any subset of J. Assume that $f \in C(J) \cap L_1(J)$, if J is bounded, or $f \in C(J) \cap L_\infty(J)$, in case J is unbounded. If f is locally* Lipα *on E, then*

$$|(L_n f)(x) - f(x)| \leq M_f(\mu_{n,2}^{\frac{\alpha}{2}}(x) + 2(d(x, E))^\alpha).$$

Proof. By the continuity of f, it is obvious that (5) holds for any $x \in J$ and $y \in \overline{E}$ (the last being the closure of the set E). Let $(x, x_0) \in J \times \overline{E}$ be so that $|x - x_0| = d(x, E)$. On the other hand, we can write

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|e_0$$

and applying the linear and positive operator L_n we obtain

$$\begin{aligned} |(L_n f)(x) - f(x)| &\leq L_n(|f - f(x_0)|, x) + |f(x) - f(x_0)| \\ &\leq L_n(M_f|e_1 - x_0|^\alpha, x) + M_f|x - x_0|^\alpha. \end{aligned} \tag{8}$$

At this point we use the classical inequality

$$(a + b)^\alpha \leq a^\alpha + b^\alpha, \quad a \geq 0,\ b \geq 0,\ 0 < \alpha \leq 1, \tag{9}$$

which implies $|t - x_0|^\alpha \leq |t - x|^\alpha + |x - x_0|^\alpha$, $t \in J$, and consequently

$$\begin{aligned} L_n(M_f|e_1 - x_0|^\alpha, x) &\leq M_f L_n(|e_1 - x|^\alpha, x) + M_f|x - x_0|^\alpha \\ &\leq M_f(\mu_{n,2}^{\frac{\alpha}{2}}(x) + |x - x_0|^\alpha). \end{aligned}$$

In the last inequality we have used estimate (7). Returning now to (8) we obtain the desired result. The proof is complete.

Define $\omega_1(f, t) := \sup\limits_{0 \leq h \leq t} \{|f(x + h) - f(x)| : \ x, x + h \in J\}$.

As a particular case of Theorem 1, when $E = J$, the following is true.

Corollary. *Let L_n be given by* (4) *and $0 < \alpha \leq 1$. If f satisfies the condition $\omega_1(f, t) = O(t^\alpha)$, then there exists a constant M_f, independent of n and x, so that $|(L_n f)(x) - f(x)| \leq M_f \mu_{n,2}^{\frac{\alpha}{2}}(x)$, $x \in J$.*

Examples.

1° Choose $J = [0, \infty)$, $I_n = \mathbb{N}$, and

$$u_{n,k}(x) = \binom{n + k - 1}{k} x^k (1 + x)^{-n-k}, \quad \omega_{n,k}(t) = (n - 1)u_{n,k}(t).$$

Then L_n reduces to the known Baskakov – Durrmeyer operator V_n (see [7]). In this case $\mu_{n,2}(x) = 2((n + 3)(x^2 + x) + 1)/(n - 2)(n - 3)$, $n \geq 4$, and $\varphi(x) = 2x(x + 1)$.

2° Choose $J = [0, \infty)$, $I_n = \mathbb{N}$, and $u_{n,k}(x) = e^{-nx}(nx)^k/k!$, $\omega_{n,k}(t) = n u_{n,k}(t)$. L_n becomes the Szasz – Durrmeyer operator S_n defined by Mazhar and Totik [6]. Now $\mu_{n,2}(x) = 2(xn^{-1} + n^{-2})$ and $\varphi(x) = 2x$.

3° If $J = [0,1]$, $I_n = \{0,1,\dots,n\}$, $u_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$, $\omega_{n,k}(t) = (n+1)u_{n,k}(t)$, then L_n becomes the Bernstein – Durrmeyer operator M_n studied by Derriennic [3]. Here $\varphi(x) = 2x(1-x)$.

4° If $J = [0,1]$, $I_n = \mathbb{N}$, $u_{n,k}(x) = \binom{n+k}{k} x^k(1-x)^{n+1}$, $\omega_{n,k}(t) = \dfrac{(n+k+1)(n+k+2)}{n+1} u_{n,k}(t)$, then L_n becomes the modified operator of Meyer – König and Zeller, studied in [8]. In this case $\varphi(x) = 2x(1-x)^2$.

3. An inverse theorem

In this section we are going to discuss an inverse result. For this purpose we shall consider the following assumptions on the sequence of operators:

(A) The operators L_n commute, i.e., $L_n \circ L_m = L_m \circ L_n$.

(B) The function φ is monotone and satisfies the relation

$$\varphi(x) \le a_0 + a_1 x + a_2 x^2, \quad x \in J. \tag{10}$$

For any given $0 < \alpha < 1$ and $E \subset J$, define

$$s_n(\alpha,t) := \left(\frac{\varphi(t)}{n}\right)^{\frac{\alpha}{2}} + n^{-\alpha} + (d(t,E))^{\alpha}, \quad t \in J.$$

(C) If h has the property $|h(t)| \le M_0 s_n(\alpha,t)$, $t \in J$, then

$$\left|\frac{d}{dt}(L_n h)(t)\right| \le M_0' \min\left\{\sqrt{\frac{n}{\varphi(t)}}, n\right\} s_n(\alpha,t), \quad t \in J, \tag{11}$$

where M_0, M_0' are independent of h, n and x. Implicitly, the functions $u_{n,k}$ belong to $C^1(J)$.

Theorem 2. *Assume that the operator* (4) *satisfies conditions* (A), (B), (C). *Let $f \in C(J) \cap L_1(J)$, if J is bounded, or $f \in C(J) \cap L_\infty(J)$ in case J is unbounded interval. If*

$$|(L_n f)(x) - f(x)| \le M_f'(\mu_{n,2}^{\frac{\alpha}{2}}(x) + (d(E,x))^{\alpha}), \quad x \in J, \tag{12}$$

then f is locally Lipα *on E.*

Proof. Let us fix an arbitrarily $(x,y) \in J \times E$. We have to show that (5) holds with a certain constant M_f. To do this we first assume that $|x-y| \geq \frac{1}{2}$. Then clearly $1 \leq 2|x-y|^\alpha$ and thus

$$|f(x) - f(y)| \leq 2\|f\|_\infty \leq 4\|f\|_\infty |x-y|^\alpha,$$

where $\|f\|_\infty = \operatorname{ess\,sup}_J f$. Therefore (5) holds in this case with $M_f = \|f\|_\infty$.

Assume now that $0 < |x-y| < \frac{1}{2}$ and define the sequence $(\delta(n,x,y))_{n\geq 1}$, $\delta(n,x,y) := \max\left\{2^{-n}, \sqrt{2^{-n}\varphi(x)}, \sqrt{2^{-n}\varphi(y)}\right\}$ which is decreasing to zero as n tends to infinity and satisfies $\delta(n,x,y) < \delta(n-1,x,y) \leq 2\delta(n,x,y)$. We can choose $n \geq 2$ so that

$$\frac{|x-y|}{2} < \delta(n,x,y) \leq |x-y|. \tag{13}$$

The following inequalities hold:

$$\left(\frac{\varphi(x)}{2^n}\right)^{\frac{\alpha}{2}} \leq \delta^\alpha(2^n,x,y) < |x-y|^\alpha, \quad 2^{-n\alpha} \leq \delta^\alpha(2^n,x,y) < |x-y|^\alpha. \tag{14}$$

We can write

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - (L_{2^n}f)(x)| + |L_{2^n}(f - L_{2^{n-1}}f, x)| \\
&\quad + |L_{2^n}(L_{2^{n-1}}f, x) - L_{2^n}(L_{2^{n-1}}f, y)| \\
&\quad + |L_{2^n}(L_{2^{n-1}}f - f, y)| + |(L_{2^n}f)(y) - f(y)| \\
&:= A_1 + B_1 + C + B_2 + A_2.
\end{aligned}$$

Estimating the terms in the last sum we shall denote by M a constant which may be different at each occurrence.

At the first step we estimate A_1 and A_2.

The relations (6), (9), (14) imply

$$\mu_{n,2}^{\frac{\alpha}{2}}(x) \leq \left(\frac{\varphi(x)}{n}\right)^{\frac{\alpha}{2}} + \lambda n^{-\alpha}, \quad \mu_{2^n,2}^{\frac{\alpha}{2}}(x) \leq (1+\lambda)|x-y|^\alpha, \tag{15}$$

where $\lambda := l \max_{i=\overline{1,l}}\{1, \|\varphi_i\|_\infty\}$. Then, making use of (12), we get

$$\begin{aligned}
A_1 &= |f(x) - (L_{2^n}f)(x)| \leq M_f'(\mu_{2^n,2}^{\frac{\alpha}{2}}(x) + d^\alpha(E,x)) \\
&\leq M_f'(2+\lambda)|x-y|^\alpha = M|x-y|^\alpha.
\end{aligned}$$

Since $d(y,E)=0$, analogously we have

$$A_2 \le M'_f(1+\lambda)|x-y|^\alpha = M|x-y|^\alpha.$$

Next we estimate B_1 and B_2.

It follows from (10) that $\varphi(t) \le \varphi(x)+a_1(t-x)+a_2\{(t-x)^2+2x(t-x)\}$. Using again (9) and (14), together with (7) and (15), we have

$$\text{(16)}\quad \begin{aligned} L_{2^n}\left(\left(\frac{\varphi}{2^{n-1}}\right)^{\frac{\alpha}{2}}, x\right) &\le 2^{\frac{\alpha}{2}} a\{|x-y|^\alpha + 2|x-y|^{\frac{\alpha}{2}}\mu_{2^n,2}^{\frac{\alpha}{4}}(x) + \mu_{2^n,2}^{\frac{\alpha}{2}}(x)\} \\ &\le M|x-y|^\alpha, \end{aligned}$$

where a is a constant which depends on a_1, a_2 and α.

Further, we have $L_{2^n}(\mu_{2^n,2},t) \le M'_f(\mu_{2^n,2}^{\frac{\alpha}{2}}(t)+d^\alpha(t,E)) + \mu_{2^n,2}(t)$ and $d^\alpha(t,E) \le d^\alpha(x,E)+|t-x|^\alpha$. Finally,

$$\begin{aligned} B_1 &\le L_{2^n}(|f-L_{2^{n-1}}f|,x) \\ &\le |f(x)-(L_{2^{n-1}}f)(x)| + M'_f L_{2^n}(\mu_{2^{n-1},2}^{\frac{\alpha}{2}} + d^\alpha(\cdot,E),x) \\ &\le M'_f\{\mu_{2^{n-1},2}^{\frac{\alpha}{2}}(x)+d^\alpha(x,E)\} \\ &\quad + M'_f L_{2^n}\left(\left(\frac{\varphi}{2^{n-1}}\right)^{\frac{\alpha}{2}} + \lambda 2^{(1-n)\alpha} + d^\alpha(x,E) + |e_1-x|^\alpha, x\right) \\ &\le M'_f\{(1+\lambda)|x-y|^\alpha + |x-y|^\alpha\} \\ &\quad + M'_f\left\{L_{2^n}\left(\left(\frac{\varphi}{2^{n-1}}\right)^{\frac{\alpha}{2}}, x\right) + 2^\alpha\lambda|x-y|^\alpha + d^\alpha(x,E) + \mu_{2^n,2}^{\frac{\alpha}{2}}(x)\right\} \\ &\le M|x-y|^\alpha. \end{aligned}$$

The relation (16) was used here. Similarly, $B_2 \le M|x-y|^\alpha$.

It remains to estimate C. In order to do this, note first that the commutativity of L_n implies

$$\begin{aligned} L_{2^n}(L_{2^{n-1}}f,t) &= \sum_{j=3}^{n}(L_{2^j}(L_{2^{j-1}}f,t) - L_{2^{j-1}}(L_{2^{j-2}}f,t)) + L_4(L_2f,t) \\ &= \sum_{j=3}^{n} L_{2^{j-1}}(L_{2^j}f - L_{2^{j-2}}f,t) + L_4(L_2f,t). \end{aligned}$$

We can write

$$\begin{aligned} &L_{2^n}(L_{2^{n-1}}f,x) - L_{2^n}(L_{2^{n-1}}f,y) \\ &\quad = \sum_{j=3}^{n}\int_y^x L'_{2^{j-1}}(L_{2^j}f - L_{2^{j-2}}f,t)\,dt + \int_y^x L'_y(L_2f,t)\,dt, \end{aligned}$$

and consequently,

$$\begin{aligned} C &:= |L_{2^n}(L_{2^{n-1}}f,x) - L_{2^n}(L_{2^{n-1}}f,y)| \\ &\leq \sum_{j=3}^{n}\left|\int_y^x L'_{2^{j-1}}(L_{2^j}f - L_{2^{j-2}}f,t)\,dt\right| + \left|\int_y^x L'_4(L_2f,t)\,dt\right|. \end{aligned}$$

But $\left|\int_y^x L'_4(L_2f,t)\,dt\right| \leq |x-y|M'\|f\|_\infty < M|x-y|^\alpha$. Set

$$c_j := \left|\int_y^x L'_{2^{j-1}}(L_{2^j}f - L_{2^{j-2}}f,t)\,dt\right|.$$

To finish the proof of the theorem we need only show that

$$\sum_{j=3}^{n} c_j \leq M|x-y|^\alpha. \tag{17}$$

Indeed, then we would have

$$|f(x)-f(y)| \leq A_1 + B_1 + C + B_2 + A_2 \leq M|x-y|^\alpha, \quad (x,y)\in J\times E$$

and thus the theorem.

Next we prove (17). Clearly,

$$\begin{aligned} |(L_{2^j}f)(t) - (L_{2^{j-2}}f)(t)| &\leq |(L_{2^j}f)(t) - f(t)| + |(L_{2^{j-2}}f)(t) - f(t)| \\ &\leq M'_f\left(\left(\frac{\varphi(t)}{2^j}\right)^{\frac{\alpha}{2}} + \lambda(2^j)^{-\alpha} + d^\alpha(t,E)\right) \\ &\quad + M'_f\left(\left(\frac{\varphi(t)}{2^{j-2}}\right)^{\frac{\alpha}{2}} + \lambda(2^{j-2})^{-\alpha} + d^\alpha(t,E)\right) \\ &\leq M\left(\left(\frac{\varphi(t)}{2^{j-1}}\right)^{\frac{\alpha}{2}} + (2^{j-1})^{-\alpha} + d^\alpha(t,E)\right) \\ &= Ms_{2^{j-1}}(\alpha,t), \end{aligned}$$

where $s_n(\alpha,t)$ was defined in the hypothesis (C). Notice that the function $h := M^{-1}|L_{2^j}f - L_{2^{j-2}}f|$ satisfies the condition required in the same hypothesis, thus we can apply (11).

CASE 1. $\delta(n,x,y) = 2^{-n}$. In view of (13), $2^n < 2|x-y|^{-1}$. Then (11) implies

$$
\begin{aligned}
c_j &\le \left| \int_y^x (L'_{2^{j-1}} h)(t) \right| \\
&\le M'_0 2^{j-1} \left\{ \left| \int_y^x \left(\frac{\varphi(t)}{2^{j-1}} \right)^{\frac{\alpha}{2}} dt \right| + (2^{j-1})^{-\alpha} |x-y| + |x-y|^{\alpha+1} \right\}.
\end{aligned}
\tag{18}
$$

Here we have deduced $d^\alpha(t,E) \le |x-y|^\alpha$ since t lies between x and y. But

$$
\begin{aligned}
\left| \int_y^x (\varphi(t))^{\frac{\alpha}{2}} dt \right| &\le |x-y| \left(\max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} \right)^\alpha \\
&\le |x-y| 2^{n\alpha/2} \delta^\alpha(n,x,y) = 2^{-n\alpha/2} |x-y|.
\end{aligned}
$$

Making use of the inequality

$$
a^2 + a^3 + \ldots + a^{n-1} < \frac{a^n}{a-1}, \quad a > 1, \tag{19}
$$

for a equal to $2^{1-\frac{\alpha}{2}}, 2^{1-\alpha}, 2$, and taking into account the relations (18), (14), we obtain

$$
\begin{aligned}
\sum_{j=3}^n c_j &\le M'_0 2^{-n\frac{\alpha}{2}} |x-y| \sum_{j=3}^n (2^{j-1})^{1-\frac{\alpha}{2}} \\
&\quad + M'_0 |x-y| \sum_{j=3}^n (2^{j-1})^{1-\alpha} + M'_0 |x-y|^{\alpha+1} \sum_{j=3}^n 2^{j-1} \\
&\le M|x-y|^\alpha.
\end{aligned}
$$

CASE 2. $\delta(n,x,y) = \max\left\{ \sqrt{\varphi(x) 2^{-n}}, \sqrt{\varphi(y) 2^{-n}} \right\}$. Consequently,

$$
\begin{aligned}
&\frac{|x-y|}{2} < \max\left\{ \sqrt{\varphi(x) 2^{-n}}, \sqrt{\varphi(y) 2^{-n}} \right\} \le |x-y| < \frac{1}{2} \\
&\left(\max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} \right)^{-1} < \frac{2^{1-n/2}}{|x-y|}.
\end{aligned}
\tag{20}
$$

Firstly we consider $\alpha \in (0, 1/2)$. This time (11) implies

$$
\begin{aligned}
c_j &\le \left| \int_y^x (L'_{2^{j-1}} h)(t) \right| \\
&\le M'_0 \left| \int_y^x \sqrt{\frac{2^{j-1}}{\varphi(t)}} \left\{ \left(\frac{\varphi(t)}{2^{j-1}} \right)^{\frac{\alpha}{2}} + (2^{j-1})^{-\alpha} + d^\alpha(t, E) \right\} dt \right| \\
&\le M_1 \left\{ (2^{j-1})^{(1-\alpha)/2} \left| \int_y^x (\varphi(t))^{\frac{\alpha-1}{2}} dt \right| \right. \\
&\quad \left. + \sqrt{2^{j-1}} ((2^{j-1})^{-\alpha} + |x-y|^\alpha) \left| \int_y^x \varphi^{-1/2}(t)\, dt \right| \right\} \qquad (21) \\
&\le M_2 \left\{ |x-y| (2^{j-1})^{\frac{1-\alpha}{2}} \left(\max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} \right)^{\alpha-1} \right. \\
&\quad + |x-y| ((2^{j-1})^{1/2-\alpha} + \sqrt{2^{j-1}} |x-y|^\alpha) \\
&\quad \left. \times \left(\max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} \right)^{-1} \right\}.
\end{aligned}
$$

With the help of (19) (we take the constant a respectively equal to $2^{(1-\alpha)/2}, 2^{1/2-\alpha}, \sqrt{2}$) and under the hypothesis of this case, we get $\sum_{j=3}^{n} c_j \le M|x-y|^\alpha$.

Now consider the case $\alpha \in [1/2, 1)$. For the given integer j, $3 \le j \le n$, there are two possibilities:

(i) $2^{j/2} \max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} < 1$. Then (20) implies $|x-y| \le 2^{-j+1}$. Using the assumption (11) we obtain

$$
\begin{aligned}
c_j &\le M'_0 2^{j-1} \left\{ \left| \int_y^x \left(\frac{\varphi(t)}{2^{j-1}} \right)^{\frac{\alpha}{2}} dt \right| + (2^{j-1})^{-\alpha} |x-y| + |x-y|^{\alpha+1} \right\} \\
&\le M'_0 \left\{ (2^{j-1})^{1-\frac{\alpha}{2}} \left(\max\left\{ \sqrt{\varphi(x)}, \sqrt{\varphi(y)} \right\} \right)^\alpha |x-y| \right. \\
&\quad \left. + |x-y| (2^{j-1})^{1-\alpha} + 2^{j-1} |x-y|^{\alpha+1} \right\}
\end{aligned}
$$

$$\leq M_0'\Big\{2^{1-\frac{\alpha}{2}}\left(\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\right)^{\alpha-1}(2^{j-1})^{\frac{1-\alpha}{2}}|x-y|$$

$$+|x-y|^\alpha(2^{j-1})^{1-\alpha}+2^{j-1}|x-y|^{\alpha+1}\Big\}$$

$$\leq M_0''|x-y|(2^{j-1})^{\frac{1-\alpha}{2}}\left(\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\right)^{\alpha-1}$$

$$+M_0'''|x-y|^{\alpha+1}(2^{j-1})^{1/2}\left(\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\right)^{-1}.$$

(ii) $2^{j/2}\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\geq 1$. Then clearly $|x-y|>2^{-j+1}$ and starting from inequality (21) we proceed further as follows

$$\begin{aligned}c_j &\leq M_3|x-y|^{1+\alpha}2^{(j-1)/2}\left(\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\right)^{-1}\\ &\quad+M_3|x-y|(2^{j-1})^{\frac{1-\alpha}{2}}\left(\max\left\{\sqrt{\varphi(x)},\sqrt{\varphi(y)}\right\}\right)^{\alpha-1}\\ &:= M_3\phi_j(\alpha,x,y).\end{aligned}$$

Taking into account the results in both situations, we obtain $c_j\leq M_4\phi_j(\alpha,x,y)$ for any $j=\overline{3,n}$. Applying the same estimates as in the previous case, we get

$$\begin{aligned}\sum_{j=3}^{n}c_j &\leq M_5|x-y|^{1+\alpha}\left(\max\left\{\sqrt{\varphi(x)2^{-n}},\sqrt{\varphi(y)2^{-n}}\right\}\right)^{-1}\\ &\quad+M_5|x-y|\left(\max\left\{\sqrt{\varphi(x)2^{-n}},\sqrt{\varphi(y)2^{-n}}\right\}\right)^{\alpha-1}\leq M|x-y|^\alpha.\end{aligned}$$

The proof of the theorem is complete.

Remark. Note that for Baskakov – Durrmeyer operator the conditions in Theorem 2 have been already verified: for (A) see [4], for (C) see [5], Lemma 2.3 and Lemma 2.4. This way our theorems lead to the result obtained by Song Li in [5].

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