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More than a Summing up about Meyer-König and Zeller Operators

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1 Introduction

A first purpose of this paper is to bring together the most outstanding results concerning the Meyer-König and Zeller operators. Furthermore, a generalization of these operators is presented establishing the degree of approximation in terms of the moduli of smoothness of first and second order. We also give a local approximation theorem by using the Ditzian-Totik modulus $\overline{\omega}_\varphi(f,\delta)$ and the weighted K-functional of second order.

Throughout the paper we use the following notations. If I is a real interval, as usually, we shall denote by $B(I)$ the Banach space of all real-valued bounded functions defined on I endowed with the sup-norm $\|\cdot\|$ defined by $\|f\| = \sup_{x\in I}|f(x)|$. $C(I)$ indicates the space of all real-valued continuous functions on I and $C^{(p)}(I)$, $p \in \mathbb{N}$, represents the linear subspace of $C(I)$ containing all p-times continuously differentiable functions. Moreover, we set $C_B(I) := C(I) \cap B(I)$. Endowed with the sup-norm, this space is also a Banach space. For $I = [0,1]$, let $K^{(p)}(I,x)$ be the class of all functions $f \in B(I)$, continuous to the left at $t = 1$ and p-times differentiable at $x \in I$. Also we denote by e_i the i-th monomial, i.e., $e_i(t) = t^i$, $i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $t \in I$.

W. Meyer-König and K. Zeller [13] introduced a sequence of linear and positive operators which were slightly modified by E.W. Cheney and A. Sharma [7] as follows:

$$(1)\qquad M_n : B([0,1)) \to C([0,1)),\quad (M_nf)(x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right),\quad n \in \mathbb{N},$$

where $m_{n,k}(x) = \binom{n+k}{k}(1-x)^{n+1}x^k$.

Actually, in the original operators of Meyer-König and Zeller the knots $k/(n+k+1)$ were used instead of $k/(n+k)$, $k \in \mathbb{N}_0$.

If $f \in B([0,1))$ is continuous from the left at $t = 1$ then $(M_nf)(1)$ is defined as (see [7])

$$(M_nf)(1) = \lim_{x\to 1^-}(M_nf)(x) = f(1).$$

M.W. Müller [14] proved that on $[0,1]$, for each $n \in \mathbb{N}$,

$$(2)\qquad (M_ne_i)(x) = e_i(x),\quad i \in \{0,1\},\quad \text{and}\quad 0 \le (M_ne_2)(x) - x^2 \le \frac{x(1-x)}{n+1}.$$

We can say that M_n has the exactness degree 1 and that it is a positive endomorphism of $C([0,1])$ with the norm $\|M_n\| = \sup\{\|M_nf\| : f \in C([0,1]),\ \|f\| \le 1\} = 1$. Also, by using the well-known theorem of Bohman-Korovkin from (2) $\lim_{n\to\infty}(M_nf)(x) = f(x)$ results for every $f \in C([0,1])$, uniformly on $[0,1]$.

2 Previous results concerning M_n operators

At first we present estimates of $M_ne_r - e_r := \Delta_{n,r}$, $r \in \mathrm{N}$, of particular interest being the case $r = 2$. In 1970 A. Lupaş and M.W. Müller [12], P.C. Sikkema [16] obtained for every $x \in [0,1]$

$$\Delta_{n,2}(x) = \frac{x(1-x)^2}{n} + o\left(\frac{1}{n}\right), \quad (n \to \infty), \quad 0 \le \Delta_{n,2}(x) \le \frac{x(1-x)^2}{n+1} + \frac{x^2(1-x)(2-x)}{(n+1)^2},$$

respectively.

In 1978 M. Becker and J. Nessel [6] proved for every $x \in [0,1]$ and $n \in \mathrm{N}$

$$\frac{x(1-x)^2}{n+1}\left(1+\frac{2x}{n+2}\right) \le \Delta_{n,2}(x) \le \frac{x(1-x)^2}{n+1}\left(1+\frac{2x}{n+1}\right). \tag{3}$$

In 1984 J.A. H. Alkemade [4] succeeded in deriving an explicit expression for the second moment in terms of a hypergeometric series, namely

$$\Delta_{n,2}(x) = x(1-x)^2\sum_{n=0}^{\infty}\frac{x^n}{\binom{2n+1}{n}} = \frac{x(1-x)^2}{n+1}\ {}_2F_1(1,2;n+2,x), \quad x \in [0,1).$$

In 1995 Ulrich Abel [1] obtained the complete asymptotic expansion for the moments M_ne_r as follows:

$$(M_ne_r)(x) \sim x^r + \sum_{k=1}^{\infty} c_k^{[r]}(x)n^{-k}, \quad (n \to \infty), \quad x \in [0,1],$$

where $c_k^{[r]}(x) = \sum_{j=1}^{r}\binom{r}{j}(-1)^jH(j-1,k+j-1,x)$ and $H(j,m,x) = \sum_{i=j}^{m} S_i^j\sigma_m^i(1-x)^{i+1}$, $(0 \le j \le m)$. The quantities S_j^i and σ_j^i denote the Stirling numbers of the first and second kind defined by $x^{(j)} = \sum_{i=0}^{j} S_j^ix^i$ and $x^j = \sum_{i=0}^{j}\sigma_j^ix^{(i)}$, $j \in \mathrm{N}_0$.
Here $x^{(k)} = x(x-1)\dots(x-k+1)$ is the falling factorial.

Further, with the help of moduli of continuity of first and second order we give the order of approximation of the M_n operators presenting local and global theorems.

1° If $f \in C([0,1])$ then we have ([12], Corollary 2.3, Theorem 2.7)

$$\|M_nf - f\| \le \frac{31}{27}\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \quad \|M_nf - f\| \le \frac{19}{9}\omega_2\left(f;\frac{1}{[\sqrt{n}]}\right), \quad n \in \mathrm{N}.$$

2° If $f \in C([0,1])$ and $x \in [0,1]$ then we have ([16], Theorem 5 and 6)

(i) $|M_nf)(x)-f(x)| \leq \left(1+x(1-x)^2+\dfrac{x^2(1-x)(2-x)}{n+1}\right)\omega_1\left(f;\dfrac{1}{\sqrt{n+1}}\right)$, $n \in$ N;

(ii) $|(M_1f)(x)-f(x)| < 1.1113\,\omega_1(f;1)$ and

$$|(M_nf)(x)-f(x)| < \left\{1+\frac{4}{27}\left(1-\frac{n^2-5}{4(n^2-1)^2}\right)\right\}\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \quad n=2,3,\ldots.$$

3° If $f \in C^{(1)}([0,1])$ then $\|M_nf-f\| \leq \dfrac{2\left(2+3\sqrt{3}\right)}{27\sqrt{n}}\omega_1\left(f';\dfrac{1}{\sqrt{n}}\right)$, $n \in$ N.
([12], Corollary 2.5).

In 1983 necessary and sufficient conditions were given by V. Totik [17] for a function to have uniform approximation or a Lipschitz order of approximation by Meyer-König and Zeller operators. For $f \in C_B([0,1))$ the following are equivalent ([17], Theorems 3 and 4)

(i) $M_nf-f=o(1)$, $(n \to \infty)$,

(ii) $\Delta^2_{h(1-x)}(f;x)=o(1)$ as $h \to 0^+$ uniformly in $x \in [0,1)$,

(iii) $f(x)-f(x+h(1-x))=o(1)$ as $h \to 0^+$ uniformly in $x \in [0,1)$,

(iv) the function $x \mapsto f\left(\dfrac{e^x}{1+e^x}\right)$ is uniformly continuous on $[0,\infty)$.

Also, if $\alpha \in (0,1]$ and $f \in C_B([0,1))$ then

(i) $M_nf-f=\mathcal{O}(n^{-\alpha})$ and

(ii) $x^\alpha(1-x)^{2\alpha}|\Delta^2_h(f;x)| \leq Kh^{2\alpha}$, $0<h\leq \dfrac{\sqrt{x}(1-x)}{4}$, $x \in [0,1)$,

are equivalent. Here $\Delta^2_h(f;x)=f(x)-2f(x+h)+f(x+2h)$, $(x,h \geq 0)$.

Regarding the asymptotic expansion for the M_n operators we point out the following three stages.

Lupaş and Müller [12] obtained the Voronovskaja-type result

$$(M_nf)(\xi)=f(\xi)+\frac{\xi(1-\xi)^2}{2n}f''(\xi)+o\left(\frac{1}{n}\right) \quad (n\to\infty), \quad f \in K^{(2)}([0,1],\xi).$$

Sikkema [16] extended this result for all $f \in K^{(4)}([0,1],\xi)$ by writing

$$(M_nf)(\xi)=f(\xi)+\frac{\xi(1-\xi)^2}{2n}f''(\xi)+\frac{1}{24n^2}\{12\xi(1-\xi)^2(2\xi-1)f''(\xi)+4\xi(1-\xi)^3(1-5\xi)f^{(3)}(\xi)$$
$$+3\xi^2(1-\xi)^4f^{(4)}(\xi)\}+o(n^{-2}) \quad (n\to\infty),$$

and remarked that $o(n^{-2})$ can be replaced by $\mathcal{O}(n^{-3})$ if f possesses at ξ even a finite sixth derivative.

In 1997 Ulrich Abel [2] gave the complete asymptotic expansion for M_n.

For $q \geq 2$ even and $f \in K^{(q)}([0,1];x)$ the following asymptotic relation holds:

$$(M_n f)(x) = f(x) + \sum_{p=2}^{q} \frac{f^{(p)}(x)}{p!}(1-x)^p b(n,p,q;x) + o(n^{-q/2}) \quad (n \to \infty),$$

where

$$b(n,p,q;x) = \sum_{k=[(p+1)/2]}^{q/2} n^{-k} \sum_{i=0}^{k} S(p,k,i)(1-x)^i$$

and

$$S(p,k,i) = \sum_{j=1}^{p} \binom{p}{j} (-1)^j S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1}.$$

Furthermore, if $f \in K^{(q+2)}([0,1],x)$, the term $o(n^{-q/2})$ can be replaced by $\mathcal{O}(n^{-(q/2+1)})$.

It is known that $[x_1, x_2, \dots, x_{n+2}; f]$ stands for the $(n+1)$-order divided difference of the function $f : I \to \mathrm{R}$ on the points $x_i \in I$, $i = 1, \dots, n+2$.

In 1944 Tiberiu Popoviciu [15] introduced the notion of higher-order convex functions. A real function f is called convex, non-concave, polynomial, non-convex and concave, respectively, of n-order on I if the divided differences of order $n+1$ are positive, non-negative, zero, non-positive and negative, respectively, on any system of $n+2$ knots from I.

After some elementary operations for $f \in C([0,1])$ it easily follows that

$$(M_{n+1} - M_n)(f) = -\sum_{k=0}^{\infty} \frac{m_{n,k}}{(k+n+1)(k+n+2)} \left[\frac{k}{k+n+1}, \frac{k+1}{k+n+2}, \frac{k+1}{k+n+1}; f \right].$$

According to the above relations (see [12], Theorem 3.3), on the subspaces of $C([0,1])$ which are formed by convex, non-concave, polynomial, non-convex or concave functions of the first order on $[0,1]$, the sequence $(M_n f)_{n \geq 1}$ is decreasing, non-increasing, stationary, non-decreasing or increasing.

The converse statement is also true: if for $f \in C([0,1])$ the sequence $(M_n f)_{n \geq 1}$ is non-increasing (non-decreasing) then the function f is non-concave (non-convex) of the first order on $[0,1]$. Moreover, the operator M_n maps convex functions of order j, $j \in \{-1, 0, 1, 2\}$, into functions which are convex of the same order.

We close this section presenting an interesting connection established by Totik [17] between M_n and the operators of Baskakov. Let $\sigma : [0,1) \to [0,\infty)$ be given by $\sigma(x) = x/(1-x)$. We have $\sigma^{-1}(y) = y/(1+y)$ and we consider $g = f \circ \sigma^{-1}$. If $f \in C_B([0,1))$ it is clear that $g \in C_B([0,\infty))$. An easy calculation gives $(M_n f)(x) = (V_{n+1}^* g)(\sigma(x))$, $n \in \mathrm{N}$, where

$$(V_n^* g)(x) = \sum_{k=0}^{\infty} b_{n,k}(x) g\left(\frac{k}{n-1}\right), \; n \geq 2, \text{ and } b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(x+1)^{n+k}}, \; x \geq 0,$$

represent the Baskakov operators with slightly modified knots.

3 A generalization of M_n operators

Over the years many authors constructed and studied approximation processes which generalize the Meyer-König and Zeller operators, the most useful in approximation theory being those in integral form both in the Kantorovich and Durrmeyer sense. Further, we propose a discrete generalization of M_n operators which depends on three sequences [3].

We define three sequences of real numbers $(\alpha_n), (\beta_n), (\gamma_{n,k})$, $(n,k) \in \mathrm{N} \times \mathrm{N}$, having the following properties:

$$1 \le \alpha_n = 1 + \mathcal{O}\left(\frac{1}{n}\right); \quad 0 \le \beta_k \le \beta_{k+1} + 1; \quad 0 \le \gamma_{n,k} \le \frac{c}{n}, \ (\forall)\ k \in \mathrm{N}, \tag{4}$$

where c is a positive constant.

Let a be a real number on the interval $(0,1)$. Assume that a sequence of functions (φ_n) satisfies the following conditions:

1° Every function φ_n is analytic on a domain Ω where $\{z \in \mathrm{C} : \ |z| \le a\} \subset \Omega$.

2° $\varphi_n^{(0)}(0) = \varphi_n(0) > 0$ and $\varphi_n^{(k)}(0) = \dfrac{d^k}{dx^k}\varphi_n(x)\Big|_{x=0} > 0,\ k \in \mathrm{N}$.

3° $\varphi_n^{(k)}(0) = \alpha_n(k+n+\beta_k)(1+\gamma_{n,k})\varphi_n^{(k-1)}(0),\ k \in \mathrm{N}$.

Hence conditions (4) are fulfilled.

We define the sequence of operators

$$(D_n f)(x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} w_{n,k}(x) f\left(\frac{k}{k+n+\beta_k}\right), \tag{5}$$

where

$$w_{n,k}(x) = \varphi_n^{(k)}(0)\frac{x^k}{k!} \quad \text{and} \quad f \in C([0,a]).$$

We observe that D_n is a linear and positive operator. If $\beta_k = 0,\ k \in \mathrm{N}$, we reobtain a generalization of Meyer-König and Zeller operators due to Ogün Doğru [9]. If, in addition, $\varphi_n(x) = (1-x)^{-n-1}$ simple calculations show that in this case $\alpha_n = 1$, $\gamma_{n,k} = 0$ for any $(n,k) \in \mathrm{N} \times \mathrm{N}$ and D_n becomes the operator M_n defined by (1). In order to study approximation properties of this sequence, we need the following result

Theorem 1 *If the operator D_n is defined by (5), then the following relations hold:*

(i) $(D_n e_0)(x) = 1$,

(ii) $0 \le (D_n e_1)(x) - x \le x(\alpha_n - 1) + \dfrac{\alpha_n c}{n} x$,

(iii) $0 \le (D_n e_2)(x) - x^2 \le (\alpha_n^2 - 1)x^2 + \alpha_n x\left(\dfrac{2c\alpha_n x + 1}{n} + \dfrac{c(\alpha_n c x + 1)}{n^2}\right)$.

An application of the Bohman-Korovkin theorem yields

Theorem 2 *If the operator D_n is defined by (5) then*

$$\lim_{n\to\infty} \|D_n f - f\| = 0, \quad \textit{for every} \quad f \in C([0,a]).$$

Applying a quantitative Korovkin type theorem due to H.H. Gonska ([11], Theorem 2.4) and taking into account Theorem 1, after few calculations we obtain

Theorem 3 *Let D_n be given by (5). The following property*

$$|(D_nf)(x) - f(x)| \le (3 + (2 + h_n^2)x^2 + (1 + h_n^{-2})n^{-1}x)\omega_2(f; h_n) + 2h_n x\omega_1(f; h_n)$$

holds, where

(6) $$h_n = ((1 + cn^{-1})\alpha_n - 1)^{1/2}.$$

If we set $\mu_{n,s}(x) := D_n((e_1 - xe_0)^s; x)$, the s-th order central moment of our operator, then Theorem 1 leads us to the following relations

(7) $$|\mu_{n,1}(x)| \le xh_n^2 \quad \text{and} \quad \mu_{n,2}(x) \le h_n^2(h_n^2 + 2)x^2 + (h_n^2 + 1)n^{-1}x,$$

where h_n is defined by (6).

Applying some classical results concerning the linear and positive operators which reproduce the monomial e_0 (see, for example, F. Altomare [5], Theorem 5.1.2), we obtain

Theorem 4 *Let D_n be given by (5).*

(i) If $f \in C([0,a])$, then $|(D_nf)(x) - f(x)| \le 2\omega_1(f; \delta_n)$.

(ii) If $f \in C^{(1)}([0,a])$, then $|(D_nf)(x) - f(x)| \le h_n x|f'(x)| + 2\delta_n\omega_1(f'; \delta_n)$.

Here $x \in [0,a]$, $\delta_n = (3h_n^2x^2 + 2n^{-1}x)^{1/2}$ and h_n is defined by (6).

In the same paper [3] we obtained an estimation of the remainder by using the Lipschitz type maximal function and we established that our operators have the variation diminishing property.

In the sequel we present a new result, more precisely a local approximation theorem via Ditzian-Totik moduli of smoothness. To that end we need the weighted K-functional of second order for $f \in C(I)$ defined by

$$K_{2,\varphi}(f,t) := \inf_{g' \in AC_{loc}(I)} (\|f - g\| + t\|\varphi^2 g''\|), \quad t > 0, \;, \; I = [0,a],$$

where $g' \in AC_{loc}(I)$ means that g is differentiable and g' is absolutely continuous in every interval $[d_1, d_2] \subset [0,a]$. Moreover, we need the Ditzian-Totik modulus of first order

$$\overrightarrow{\omega}_\varphi(f,\delta) := \sup_{0<h\le\delta} \; \sup_{x, x+h\varphi(x) \in I} |f(x + \varphi(x)h) - f(x)|.$$

Given that $\varphi : I \to \mathrm{R}$ is an admissible step-weight function, it is well known that the K-functional $K_{2,\varphi}(f,t^2)$ and the Ditzian-Totik modulus

$$\omega_{2,\varphi}(f,t) := \sup_{0<h\le t} \; \sup_{x \pm h\varphi(x) \in I} |f(x - \varphi(x)h) - 2f(x) + f(x + \varphi(x)h)|$$

are equivalent (see [8]). Taking into account (1) we can assume (without any loss of generality) that $\alpha_n \le (3/2)n(n+c)^{-1}$. This relation together with (6) and (7) imply $|\mu_{n,1}(x)| \le 1/2$. By using a result due to M. Felten ([10], Theorem 1), our previous relations allow us to state the following proposition.

Theorem 5 *Let D_n be given by (5). If $\varphi_1, \varphi_2 : I \to \mathrm{R}$ are functions with φ_1 an admissible step-weight function of the Ditzian-Totik modulus and with φ_2^2 concave, then*

$$|(D_n f)(x) - f(x)| \leq \overline{\omega}_{\varphi_1}\left(f, \frac{xh_n^2}{\varphi_1(x)}\right) + 4K_{2,\varphi_2}\left(f, \frac{h_n^2(h_n^2+2)x^2 + (h_n^2+1)n^{-1}x}{\varphi_2^2(x)}\right)$$

holds true for $x \in [0,a]$ and $f \in C([0,a])$.

In the particular case $D_n \equiv M_n$ we have $\mu_{n,1} = 0$ and $\mu_{n,2} = M_n e_2 - e_2 = \Delta_{n,2}$. We can choose the step weight function $\varphi_2(x) = \sqrt{x}(1-x)$; φ_2^2 is concave on $[0, 2/3]$ and, by using the relation (3), we obtain

Corollary 1 *Let M_n be given by (1). Then the inequality*

$$|(M_n f)(x) - f(x)| \leq 4K_{2,\varphi_2}\left(f, \left(1 + \frac{2x}{n+1}\right)\frac{1}{n+1}\right), \quad f \in C([0,1)), \quad x \in [0, 2/3],$$

holds.

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