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On a certain class of approximation operators

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Abstract. The paper is devoted to the study of an approximation process K_n^H representing an integral form in Kantorovich sense of Bernstein-Sheffer operators.

We establish the degree of approximation both in $C[0,1]$ space in terms of the modulus of continuity and in $L_p[0,1]$, $p \geq 1$, spaces in terms of the integral modulus of smoothness.

Consequently, it results that the sequence $(K_n^H)_{n\geq 1}$ converges to the identity operator in the mentioned spaces.

Also we point out a connection between the smoothness of local Lipschitz $-\alpha$ $(0 < \alpha \leq 1)$ functions and the local approximating property.

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1 Introduction

For any positive integer n we denote by Π_n the linear space of polynomials of not most than n degree and by Π_n^* the set of all polynomials of exactly n degree.

We set $\Pi := \bigcup_{n\geq 0} \Pi_n$. A polynomial sequence $b = (b_n)_{n\geq 0}$, $b_n \in \Pi_n^*$ will be called of binomial type if for any $n \geq 0$ the following identity

$$b_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), \quad (x,y) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

holds.

The most common example is the monomials e_n, $e_n(x) = x^n$.

Let J be a linear operator applicable to all polynomials having the form

$$J(y) = c_1 y' + c_2 y'' + c_3 y''' + \dots, \quad c_1 \neq 0. \tag{2}$$

A sequence $s = (s_n)_{n\geq 0}$, $s_n \in \Pi_n^*$, is of type zero (Sheffer sequence, [10]) if $J(s_n) = s_{n-1}$, $n \geq 0$, where $s_{-1} := 0$.

The formal series $\tilde{J}(t) = \sum_{j\geq 1} c_j t^j$ is called the generating series (or function) for the operator J.

Let the formal power series inverse of $\tilde{J}$ be

$$H(t) = \sum_{n\geq 1} h_n t^n, \quad h_1 = c_1^{-1} \neq 0, \tag{3}$$

obtained from $\tilde{J}(H(t)) = H(\tilde{J}(t)) = t$.

We consider the sequence $p = (p_n)_{n\geq 0}$ generated by

$$e^{xH(t)} = \sum_{n\geq 0} p_n(x)\frac{t^n}{n!} \tag{4}$$

and by using the properties of the exponential function it can be proved that p verifies the condition (1) thus it is of binomial type.

It is known ([10], Theorem 2.1) that a necessary and sufficient condition that the sequence s be of type zero corresponding to the operator J of (2) is that a_n, $n \geq 0$, exist so that

$$A(t)e^{xH(t)} = \sum_{n\geq 0} s_n(x)\frac{t^n}{n!},$$

where $A(t) = \sum_{n\geq 0} a_n t^n$, $a_0 \neq 0$. The condition $a_0 \neq 0$ guarantees that s_n belongs to Π_n^*.

Also, we have

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)p_{n-k}(y).$$

We point out that the binomial sequences are in connection with the umbral calculus.

Let us remind that the linear operator $J : \Pi \to \Pi$ becomes a delta operator if $E^a J = JE^a$, $a \in \mathbb{R}$, and Je_1 is a non-zero constant where E^a is the shift operator $((E^a f)(x) = f(x+a))$. R. Mullin and G.C. Rota [7] had proved that every delta operator generates a unique sequence p of binomial type satisfying $p_0 = e_0$, $p_n(0) = 0$ and $Jp_n = np_{n-1}$, $n \geq 1$.

Also, the binomial sequences may be used for the construction of linear and positive approximation operators [6], [9].

The aim of this paper is to investigate an integral operator of Kantorovich type created by using a binomial sequence.

2 Construction of the operators

Starting from (4) the Bernstein-Sheffer operator of degree n, associated with the function H and consequently with the sequence p is defined by

$$(B_n^H f)(x) = \frac{1}{p_n(1)}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}p_k(x)p_{n-k}(1-x), \quad f \in C[0,1], \tag{5}$$

provided that $p_n(1) \neq 0$ for all $n \geq 1$.

The classical Bernstein operator corresponds to $H(t) = t$ which implies $p_n = e_n$.

In [2] and [9] many nice properties of B_n^H are releaved and Lupaş [6] has given a generalization $L_n^{\langle a_n \rangle}$ of these operators by using a sequence $(a_n)_{n\geq 1}$ of positive numbers.

But the above operators cannot be used for L_p $(1 \le p < \infty)$-approximation. For this purpose we must modify them into integral form.

Actually, we replace $f(k/n)$ in the formula (5) by an integral mean of $f(x)$ over a small interval around the point k/n as follows

$$(K_n^H f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad f \in L_1[0,1], \tag{6}$$

where $p_{n,k}(x) = \dfrac{1}{p_n(1)}\dbinom{n}{k} p_n(x) p_{n-k}(1-x)$.

By using (1) it is clear that

$$\sum_{k=0}^{n} p_{n,k}(x) = 1. \tag{7}$$

REMARK

(i) K_n^H is a linear operator. P. Sablonniere ([9], Theorem 1) characterized those functions H for which B_n^H is a positive operator. Actually, the basic idea appeared long time ago, in [8] T. Popoviciu, presenting the solution of this problem.

In the same paper, choosing $H(t) = t(1-t)^{-1}$ and using the Laguerre polynomials the author also constructed operators of binomial type.

However, K_n^H becomes a positive operator if and only if h_n defined by (3) are non-negative for all $n \ge 2$. In what follows we assume that this condition is always fulfilled.

(ii) If we define $\overline{H}$ by $\overline{H}(t) = H(-t)$ then the relation (4) implies $\overline{p}_n(x) = (-1)^n p_n(x)$ and (6) leads us to the fact that K_n^H and $K_n^{\overline{H}}$ coincide.

(iii) We present another look of K_n^H.

More exactly, we can write this operator as a singular integral of the type

$$(K_n^H f)(x) = \int_0^1 W_n(x,t) f(t)dt, \quad x \in [0,1],$$

with the non-negative kernel $W_n(x,t) = (n+1)p_{n,k}(x)$ for $k/(n+1) < t \le (k+1)/(n+1)$, $k = \overline{0,n}$ and $W_n(x,0) = 0$, $x \in [0,1]$. Using (7) it is obvious that our kernel satisfies

$$\int_0^1 W_n(x,t)dt = 1.$$

3 Properties of the operators

We start with the expression of the operator K_n^H on the test functions e_j, $j = \overline{0,2}$.

MMA 1 *Let $(K_n^H)_{n\ge 1}$ be defined by (6) and let the sequence $(r_n(x))_{n\ge 0}$ be nerated by $H''(t)\exp(xH(t)) = \sum_{n\ge 0} r_n(x)t^n/n!$.*

The following identities

(i) $K_n^H e_0 = e_0$

(ii) $K_n^H e_1 = \frac{n}{n+1}e_1 + \frac{1}{2(n+1)}$

(iii) $(K_n^H e_2)(x) = \frac{n}{(n+1)^2}\left\{(n-1)(1-q_n)e_2(x) + (2+(n-1)q_n)e_1(x) + \frac{1}{3n}\right\}$

old, where $q_n = r_{n-2}(1)/p_n(1)$.

roof. First, we recall some useful formulas related to Bernstein-Sheffer operators ([9], Theorem 2(ii)):

$$B_n^H e_0 = e_0, \quad B_n^H e_1 = e_1, \quad B_n^H e_2 = e_2 + \left(\frac{1}{n} + \frac{n-1}{n}q_n\right)(e_1 - e_2), \tag{8}$$

From (7) we have $(K_n^H e_0)(x) = (B_n^H e_0)(x) = e_0(x)$.
Further we deduce

$$(K_n^H e_1)(x) = \frac{n}{n+1}\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} + \frac{1}{2(n+1)}\sum_{k=0}^{n} p_{n,k}(x) =$$

$$= \frac{n}{n+1}(B_n^H e_1)(x) + \frac{1}{2(n+1)}(B_n^H e_0)(x)$$

which implies the second statement.

Also, after few calculations we obtain

$$(K_n^H e_2)(x) = \frac{1}{(n+1)^2}\left\{n^2(B_n^H e_2)(x) + n(B_n^H e_1)(x) + \frac{1}{3}\right\}$$

and with the help of (8) we arrive at the desired result. □

Choosing $H = e_1$ it results $p_n(x) = x^n$ and $q_n = 0$.

The operator $K_n^{e_1}$ is the n-th Kantorovich operator and the results of Lemma 1 become known identities (for example see [1], Section 5.3.7).

Further we need the central moments of K_n^H defined by $\Omega_{n,j}(x) := K_n^H((e_1 - xe_0)^j, x)$, $j \ge 0$.

Taking into account Lemma 1 for any $n \ge 1$ it is easy to prove that

$$\Omega_{n,0}(x) = 1, \quad \Omega_{n,1}(x) = \frac{1}{n+1}\left(\frac{1}{2} - x\right),$$
$$\Omega_{n,2}(x) = \frac{n(n-1)}{(n+1)^2}\left(\frac{1}{n} + |q_n|\right)x(1-x) + \frac{1}{3(n+1)^2}. \tag{9}$$

THEOREM 2 *Let $(K_n^H)_{n\ge 1}$ be defined by (6) such that H has positive coefficients.*

(i) *If* $f \in C[0,1]$ *then* $|(K_n^H f)(x) - f(x)| \le (3/2)\omega_f\left(\sqrt{1/n + |q_n|}\right)$ *where* ω_f *is the modulus of continuity associated to* f.

The sequence $(K_n^H f)_{n\ge 1}$ *converges uniformly to* f *if and only if* $\lim_{n\to\infty} q_n = 0$ *holds.*

(ii) *If* $f \in L_p[0,1]$, $p \ge 1$ *and* $\lim_{n\to\infty} q_n = 0$ *then* $\lim_{n\to\infty} \|K_n^H f - f\|_p = 0$.

Proof. (i) Since H has positive coefficients, $p_n(x) \ge 0$ and consequently $p_{n,k} \ge 0$, $k = \overline{0,n}$.

This fact together with Lemma 1 implies

$$|(K_n^H f)(x) - f(x)| = \left|(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (f(t) - f(x))dt\right| \le$$

$$\le (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t) - f(x)|dt.$$

We use firstly the following known properties of the modulus of continuity $|f(t) - f(x)| \le \omega_f(|t-x|) \le (1 + \delta^{-1}|x-t|)\omega_f(\delta)$ for any $\delta > 0$ and secondly Cauchy's inequality $\sum_{k=0}^{n} p_{n,k}(x)|x-t| \le \left(\sum_{k=0}^{n} p_{n,k}(x)(x-t)^2\right)^{1/2}$.

By virtue of the linearity of our operators and by Lemma 1 we obtain

$$|(K_n^H f)(x) - f(x)| \le (1 + \delta^{-1}\Omega_{n,2}^{1/2}(x))\omega_f(\delta). \tag{10}$$

Since $x(1-x) \le 1/4$, $x \in [0,1]$, from (9) we can write $\Omega_{n,2}(x) \le \frac{n-1}{4(n+1)^2}(1 + n|q_n|) + \frac{1}{3(n+1)^2} \le \frac{1}{4}(n^{-1} + |q_n|)$.

Choosing $\delta := (n^{-1} + |q_n|)^{1/2}$, from the relation (10) the conclusion follows.

The second assertion follows directly from Lemma 1 and the well-known theorem of Bohman-Korovkin. We notice that the sufficient part of the statement can also be obtained by the previous inequality of the present theorem.

(ii) By using the natural embedding $S_p : C[0,1] \to L_p[0,1]$, $S_p(f) = f$, we obtain that Korovkin subspaces in $C[0,1]$ are also Korovkin subspaces in $L_p[0,1]$ ([1], section 4.1).

Thus, Lemma 1 implies as well $\lim_{n\to\infty} K_n^H f = f$ in $L_p[0,1]$-norm. □

We are going to estimate the degree of approximation by using the r^{th} order modulus of smoothness of f measured in $L_p[0,1]$-spaces, $p \ge 1$.

We recall

$$\omega_r(f,t)_p = \sup_{0<|h|\le t} \|\Delta_h^r f\|_p, \quad f \in L_p[0,1], \quad t > 0,$$

where $\Delta_h^r f(x) = (E^h - I)^r f(x)$. For any $k \le r$, $(E^h)^k f(x) = f(x+kh)$ if $x, x+kh$ belong to $[0,1]$ and becomes zero otherwise.

For a more complete documentation [3] can be consulted.
Also $\|\cdot\|_p$ represents the usual norm in $L_p[0,1]$ spaces.
We consider the subspaces $W_{p,r}$ ($p \geq 1$, $r \geq 0$ integers) of the L_p space re the functions possess smooth derivatives.
More specifically $W_{p,r}$ stands for the space which consists of those functions n $[0,1]$ for which the first $r-1$ derivatives are absolutely continuous on $[0,1]$ l the r^{th} derivative belongs to $L_p[0,1]$.
Also we need the Peetre K-functional of $f \in L_p[0,1]$ which is defined by ([4],

$$K(t,f;X,Y) = \inf\{\|f-g\|_p + t(\|g\|_p + \|g^{(r)}\|_p : g \in Y\},$$

$:= L_p[0,1]$, $Y := W_{p,r}[0,1]$, and the modified K' functional is given by

$$K'(t,f;X,Y) = \inf\{\|f-g\|_p + t\|g^{(r)}\|_p : g \in Y\}.$$

The following connections between these functionals and the modulus of noothness are valid

$$\begin{gathered} K'(t,f;X,Y) \leq K(t,f;X,Y) \leq \min(1,t)\|f\|_p + 2K'(t,f;X,Y), \\ c_1\omega_r(f,t)_p \leq K'(t^r,f;X,Y) \leq c_2\omega_r(f,t)_p, \quad 0 < t \leq 1, \end{gathered} \tag{11}$$

here $c_1 = c_1(p,r)$, $c_2 = c_2(p,r)$ are positive constants.

THEOREM 3 *Let $(K_n^H)_{n\geq 1}$ be defined by (6) such that H has positive coefficients* *nd $\alpha_{n,p} = (n^{-1} + |q_n|)(p+1)^{-1/p} + (1/3)(n+1)^{-2}$.*

If $r \geq 3$ is an integer and $\alpha_{n,p}^{1/r} \leq (2r)^{-1}$, $n \in \mathbb{N}$, then for every $f \in L_p[0,1]$ the following inequality

$$\|K_n^H f - f\|_p \leq 2\alpha_{n,p}\|f\|_p + C_{p,r}\omega_r(f, 2r\alpha_{n,p}^{1/r})_p$$

holds, where $C_{p,r}$ is a constant independent of f and n.

Proof. Following [11], for any $(t,x) \in [0,1]\times[0,1]$ and $g \in W_{p,r}[0,1]$ We can write $g(t)-g(x) = g'(x)(t-x)+\varphi_x(t)$ where $\varphi_x(t) = \displaystyle\int_x^t (t-u)g''(u)du$. Consequently $|\varphi_x(t)| \leq (t-x)^2\|g''\|_\infty$ holds.

Applying the linear and positive operators K_n^H we get

$$|K_n^H(g-g(x)e_0,x)| \leq \|g'\|_\infty |K_n^H(e_1 - xe_0, x)| + \|g''\|_\infty K_n^H((e_1-xe_0)^2, x),$$

which implies

$$\|K_n^H(g-g(x)e_0,\cdot)\|_p \leq \|g'\|_\infty\|\Omega_{n,1}\|_p + \|g''\|_\infty\|\Omega_{n,2}\|_p. \tag{12}$$

At this point we need a known inequality: for $k = 0,1,\ldots,r-1$ and any $0 < \varepsilon \leq 1$ one has $\|g^{(k)}\|_\infty \leq \varepsilon^{1/q}\{(2r)^r\varepsilon^{-k-1}\|g\|_p + \varepsilon^{r-k-1}\|g^{(r)}\|_p\}$, where $1/q = 1-1/p$ ([5], relation (1)).

Choosing $\varepsilon = 1$ and $j \in \{1,2\}$ the following relation

$$\|g^{(j)}\|_\infty \leq (2r)^r\|g\|_p + \|g^{(r)}\|_p \tag{13}$$

holds.

Further, taking into account (9) and for the sake of simplicity denoting $\tilde{q}_n := (n-1)(1+n|q_n|)/(n+1)^2$ we obtain

$$\|\Omega_{n,1}\|_p = \frac{1}{2(n+1)}\left(\int_0^1 |1-2x|^p dx\right)^{1/p} = \frac{1}{2(n+1)(p+1)^{1/p}},$$

$$\|\Omega_{n,2}\|_p = \left(\int_0^1 |\tilde{q}_n(x-x^2)+(1/3)(n+1)^{-2}|^p dx\right)^{1/p} \le$$

$$\le \tilde{q}_n\left(\int_0^1 (x-x^2)^p dx\right)^{1/p} + (1/3)(n+1)^{-2} =$$

$$= \tilde{q}_n\left(\frac{(p!)^2}{(2p+1)!}\right)^{1/p} + \frac{1}{3(n+1)^2} \le \left(\frac{1}{n}+|q_n|\right)\frac{1}{(p+1)^{1/p}} + \frac{1}{3(n+1)^2} = \alpha_{n,p}.$$

We deduce that $\max\{\|\Omega_{n,1}\|_p, \|\Omega_{n,2}\|_p\} \le \alpha_{n,p}$ is valid.

Thus, the relations (12) and (13) imply

$$\|K_n^H g - g\|_p = \|K_n^H g - gK_n^H e_0\|_p \le 2((2r)^r\|g\|_p + \|g^{(r)}\|_p)\alpha_{n,p}.$$

For a given function $f \in L_p[0,1]$ with the help of the above relation, we can write

$$\begin{aligned}\|K_n^H f - f\|_p &\le \|K_n^H(f-g)\|_p + \|K_n^H g - g\|_p + \|g-f\|_p \le \\ &\le 2\{\|f-g\|_p + (2r)^r\alpha_{n,p}(\|g\|_p + \|g^{(r)}\|_p)\}.\end{aligned}$$

Taking the infimum over all $g \in W_{p,r}[0,1]$ and using (11) we get

$$\begin{aligned}\|K_n^H f - f\|_p &\le 2K((2r)^r\alpha_{n,p}, f; X, Y) \\ &\le 2\min(1,(2r)^r\alpha_{n,p})\|f\|_p + 4c_2\omega_r(f, 2r\alpha_{n,p}^{1/r})_p.\end{aligned}$$

Because $\alpha_{n,p}^{1/r} \le (2r)^{-1}$, the proof of our theorem is complete. □

Since $\omega_r(f,t)_p$ is a nondecreasing function in t for each f and verifies the property $\omega_r(f,mt)_p \le m^r\omega_r(f,t)_p$ for any natural m, we have

$$\omega_r(f, 2r\alpha_{n,p}^{1/r})_p \le (2r)^r\omega_r(f, |q_n|^{1/r} + (2/n)^{1/r})_p.$$

We also relied on the following increases $\alpha_{n,p} \le |q_n| + 2/n$, $(u+v)^\beta \le u^\beta + v^\beta$ ($\beta \in (0,1]$, $u \ge 0$, $v \ge 0$). So we can mark a simpler form of our result.

Corollary 4 *Let $(K_n^H)_{n\ge1}$ be given by (6) such that H has positive coefficients, $\lim_{n\to\infty} q_n = 0$ and $r \ge 3$ an integer.*

For every $f \in L_p[0,1]$ and for sufficiently large n one has

$$\|K_n^H f - f\|_p \le 2(|q_n| + 2/n)\|f\|_p + C'_{p,r}\omega_r(f, |q_n|^{1/r} + (2/n)^{1/r})_p,$$

where $C'_{p,r}$ is a constant independent of f and n.

LEMMA 5 *If* $(K_n^H)_{n\geq 1}$ *is defined by (6) and* H *has positive coefficients then*

$$K_n^H(|e_1 - xe_0|^\alpha, x) \leq \Omega_{n,2}^{\alpha/2}(x), \quad x \in [0,1], \quad 0 < \alpha \leq 1.$$

Proof. It is a direct result of Hölder's inequality. Indeed, let's take the numbers $r > 0$, $s > 0$, such that $1/r + 1/s = 1$. Hölder's inequality and the relation (7) imply $K_n^H(h^\alpha, x) \leq (K_n^H(h^{\alpha r}, x))^{1/r}$. Choosing $r = 2/\alpha$, $h = |e_1 - xe_0|$ the conclusion follows. □

In order to investigate the relationship between the local smoothness of function and the local approximation we recall that a continuous function f is locally $Lip\alpha$ $(0 < \alpha \leq 1)$ on $E \subset [0,1]$ if it satisfies the condition

$$|f(x) - f(y)| \leq M_f|x - y|^\alpha, \quad (x, y) \in [0,1] \times E, \tag{14}$$

where M_f is a constant depending only of α and f. Also we set by $d(x, E)$ the distance between x and E defined as $d(x, E) := \inf\{|x - t| : t \in E\}$.

THEOREM 6 *Let* $(K_n^H)_{n\geq 1}$ *be given by (6) such that* H *has positive coefficients,* $0 < \alpha \leq 1$ *and* E *be any subset of* $[0,1]$.

If f *is locally* $Lip\alpha$ *on* E *then one has*

$$|(K_n^H f)(x) - f(x)| \leq M_f(\beta_n(x, \alpha) + 2d^\alpha(x, E)), \quad x \in [0,1], \tag{15}$$

where $\beta_n(x, \alpha) = \{(1/n + |q_n|)x(1 - x)\}^{\alpha/2} + (n + 1)^{-\alpha}$.

Proof. By using the continuity of f it is obvious that (14) holds for any $x \in [0,1]$ and $y \in \overline{E}$, the closure of the set E.

Let $(x, x_0) \in [0,1] \times \overline{E}$ such that $|x - x_0| = d(x, E)$.

We have

$$|(K_n^H f)(x) - f(x)| \leq K_n^H(|f - f(x_0)|, x) + |f(x) - f(x_0)| \leq$$
$$\leq M_f\{K_n^H(|e_1 - x_0e_0|^\alpha, x) + |x - x_0|^\alpha\}.$$

Also we get $|t - x_0|^\alpha \leq |t - x|^\alpha + |x - x_0|^\alpha$, $t \in [0,1]$, and by Lemma 2 we have $K_n^H(|e_1 - x_0e_0|^\alpha, x) \leq \Omega_{n,2}^{\alpha/2}(x) + |x - x_0|^\alpha$.

In view of (9) we obtain

$$\Omega_{n,2}^{\alpha/2}(x) \leq \left\{\frac{n(n-1)}{(n+1)^2}(1/n + |q_n|)x(1 - x)\right\}^{\alpha/2} + 3^{-\alpha/2}(n + 1)^{-\alpha} \leq \beta_n(x, \alpha).$$

The result follows. □

In particular we can choose $E = [0,1]$ and it results the following statement.

COROLLARY 7 *Let* $(K_n^H)_{n\geq 1}$ *be given by (6) such that* H *has positive coefficients and* β_n *be defined by (15) where* $0 < \alpha \leq 1$.

If $f \in Lip\alpha$ *on* $[0,1]$ *then*

$$|(K_n^H f)(x) - f(x)| \leq M_f\beta_n(x, \alpha), \; x \in [0,1].$$

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