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à 25 ans depuis sa mort

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Application of Popoviciu's high convexity to the study of some sequences properties

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Abstract. Starting by the notion of convex functions of n-order introduced by Tiberiu Popoviciu, we aim to record those theorems obtained in time by several mathematicians, which show how the behaviour of a function f is mirrored by sequences of linear and positive operators. From the perspective opened by T. Popoviciu, this survey paper presents the monotonicity properties of the well-known operators of Bernstein, Bleimann Butzer and Hahn, Meyer König and Zeller, Szasz Favard-Mirakyan and Baskakov. This way we take the opportunity to emphasize the importance of the powerful school founded by Popoviciu in Cluj.

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1 Introduction

Brilliant mind, T. Popoviciu has had important contributions in various areas of mathematics. The study of the convex functions of high order was very close to his heart, repeatedly treating it in one way or another in his research career. He became well-known for his remarkable papers *Notes sur les fonctions convexes d'ordre supérieur*, which he started writing in 1936 and which were published in various journals. In 1944 all his contributions in this domain were gathered in the monograph [17] which will be quoted by many authors in time. We mention that in the *Introduction* to this book, Tiberiu Popoviciu has written "Dans ce petit livre, j'expose les principales propriétés et quelques généralisations des fonctions convexes d'une ou de plusieurs variables".

The aim of this note is to present the application of the above notion to the study of monotonicity properties of the linear and positive operators. More precisely, the divided differences can play a fundamental role in the investigation of the approximation properties of the classical linear and positive processes. These sequences have the remarkable characteristics of copying the fundamental features of the functions which are approximated as positivity, monotonicity or convexity of different orders.

First of all we briefly present the notion of n-convexity introduced by Tiberiu Popoviciu and then we illustrate how this fundamental mathematical tool can be used in the approximation theory. In the same time we mean to underline the author's contribution to the creation of a mathematical school in this field located in Cluj.

2 Convex function of n-order

For a function f and the set of $n+1$ points $X = \{x_0, x_1, \ldots, x_n\}$ belonging to its domain we define the n^{th} divided difference by the recurrence relation

(2.1)
$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, x_2, \ldots, x_n; f] - [x_0, x_1, \ldots, x_{n-1}; f]}{x_n - x_0},$$

where $[x; f] = f(x)$. Among the basic formulas of the divided difference of n order we recall the following (for example see [3], Section 3.1)

(i) $[x_0, x_1, \ldots, x_n; f] =$

$$= \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)} =$$

$$= \sum_{k=0}^{n} \frac{f(x_k)}{\Omega'(x_k)},$$

with

$$\Omega(x) = \prod_{j=0}^{n} (x - x_j). \tag{2.2}$$

(ii) $[x_0, x_1, \ldots, x_n; f] = \dfrac{W(x_0, x_1, \ldots, x_n; f)}{V(x_0, x_1, \ldots, x_n)}$,
where $V(x_0, x_1, \ldots, x_n)$ is the Vandermonde determinant corresponding to the numbers x_k, $k = \overline{0, n}$, and $W(x_0, x_1, \ldots, x_n; f)$ is the determinant obtained from the previous one by replacing its $(n+1)$-th column by the column of $f(x_k)$, $k = \overline{0, n}$.

It clearly results that $[x_0, x_1, \ldots, x_n; f]$ is symmetrical in $x_0, x_1, \ldots, x_n$, is a constant if f is a polynomial of degree less or equal to n and is zero for a polynomial of degree strictly less than n.

Definition 2.1 (Popoviciu, [17], p.13) *A function f is called convex, non-concave, polynomial, non-convex, respectively concave of n-order on E if the following relation*

$$[x_0, x_1, \ldots, x_{n+1}; f] > 0, \geq 0, = 0, \leq 0, \textit{resp.} < 0, \tag{2.3}$$

holds, for any $n + 2$ distinct nodes belonging to E.

For $n = 0$ we obtain the monotone functions: increasing, non-decreasing, constant, non-increasing resp. decreasing. For $n = 1$ we obtain the convex functions, non-concave, linear, non-convex, resp. concave in ordinary sense. T. Popoviciu mentioned that for $n = 1$ the above definition was already given in 1916 by L. Galvani. Also for an arbitrary n and an interval E, the functions of n-order were introduced in 1926 by E. Hopf in his doctoral thesis. The generalization of this concept on an arbitrary set E is due to Popoviciu and appeared for the first time in 1934, see [15].

Definition 2.1 is equivalent to the following.

Definition 2.2 (Popoviciu, [17], p.15) *A function f is called convex, non-concave, polynomial, non-convex, respectively concave of*

n-order on E if the following relation

$$L_n(x_0, x_1, \ldots, x_n; f|x) < f(x), \leq f(x), = f(x), \geq f(x), \textit{resp.} > f(x),$$

holds, for $x > x_n$ and $x_0, x_1, \ldots, x_n, x \in E$.

Here $L_n(x_0, x_1, \ldots, x_n; f|\cdot)$ represents the Lagrange polynomial of n order and is given by the formula

$$L_n(x_0, x_1, \ldots, x_n; f|x) = \sum_{i=0}^{n} \frac{\Omega(x) f(x_i)}{(x - x_i)\Omega'(x_i)},$$

Ω being defined by (2.2).

In [16], T. Popoviciu proved that a function f of n-order is characterized by the inequality

$$[x_0, x_1, \ldots, x_{n+1}; f][x_1, x_2, \ldots, x_{n+2}; f] \geq 0,$$

for every $x_k \in E$, $k = \overline{0, n+2}$.

In other words, a function f is not of n-order if and only if $(n+3)$-nodes exist in E such that $x_0 < x_1 < \cdots < x_{n+2}$ and $[x_0, x_1, \ldots, x_{n+1}; f][x_1, x_2, \ldots, x_{n+2}; f] < 0$.

A generalization in Jensen sense of the n-convexity can be formulated as below.

Definition 2.3 (Popoviciu, [17], p.48) *A function f defined on an interval (a, b) is called convex, non-concave, polynomial, non-convex respectively concave of n order (J) - in Jensen sense - on (a, b) if the following relation*

$$(\Delta_h^{n+1} f)(x) > 0, \geq 0, = 0, \leq 0 \ \textit{resp.} < 0,$$

$x, x + (n+1)h \in (a, b)$, $h > 0$, holds.

We recall that $(\Delta_h^r f)(x)$ is the forward r^{th} difference of the function $f : (a, b) \to \mathbf{R}$ defined by

$$(\Delta_h^r f)(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh),$$

for every $x \in (a, b)$ such that $x + rh \in (a, b)$.

Between the forward r^{th} difference and the r^{th} divided difference, $r \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, the following connection takes place:

$$(\Delta_h^r f)(x) = r! h^r [x, x+h, x+2h, \ldots, x+rh; f].$$

Obviously, every function f of n-order (J) verifies the inequality (2.3).

3 On the monotonicity properties of classical sequences of operators

This section develops an investigation of the monotonicity properties of some linear and positive operators of discrete type. This kind of study is the aim of a lot of papers and it uses as principal tool the divided differences of high orders. Most of the papers that will be quoted here mention in their references Popoviciu's work.

1. We consider the Bernstein operators B_n which are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1], \quad n \in \mathbf{N},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $f \in \mathbf{R}^{[0,1]}$.

In 1964, D.D. Stancu [18] obtained a representation of the remainder in Bernstein's approximation formula $f = B_n f + R_n f$ under the form of an average of certain divided differences of second-order, namely

$$(R_n f)(x) = -\frac{x(1-x)}{n} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n}; f\right].$$

An important consequence of this result is the following: if f is convex (concave) of first order on $[0,1]$ then we have $B_n f > f$ (respectively $B_n f < f$) on $(0,1)$. Related to the monotonicity of the sequence of Bernstein polynomials, D.D. Stancu established

[19], Theorem 1, the nice formula

$$(B_{n+1}f)(x) - (B_n f)(x) =$$

$$= -\frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right].$$

Sometimes it is inaccurately attributed to B. Averbach, e.g. in [7], Theorem 4.1, page 309.

It should be mentioned that the monotonicity properties of the sequence $(B_n f)_{n\geq 1}$ have been first studied by W.B. Temple in 1954 and later in 1957 by O. Aramă. The monotonicity properties of the sequence $(B_n f)'$, $n \in \mathbf{N}$, have been investigated by D.D. Stancu in [19] and as we will present further, the same author has extended the result to higher derivatives.

2. Considering an integer r $(0 \leq r \leq n)$ and the following knots of the interval $I = [a, b]$: $a_i = a + ih$, $i = \overline{0, n}$, $b_j = a + jl$, $j = \overline{1, n}$, where $0 < h \leq (b-a)/n$, $0 < l < (b-a)/n$, D.D. Stancu has associated [21] to each function f defined by $[a, b]$ the linear functionals $T_{n,k}^{(\nu)}$, $k = \overline{0, n}$, $\nu = \overline{1, r+1}$, defined recursively as follows

$$T_{n,k}^{(2)}(f) = [a_k, a_{k+1}, b_{k+1}; f],\ 0 \leq k \leq n-1,$$

$$T_{n,k}^{(\nu+1)}(f) = T_{n,k+1}^{(\nu)}(f) - T_{n,k}^{(\nu)}(f), \tag{3.1}$$

$$1 < \nu \leq r,\ 0 \leq k \leq n-r.$$

Also, we set $m^{[s]} = m(m-1)\ldots(m-s+1)$. One obtains

Theorem 3.1 (D.D. Stancu, [21], Th.4.1) *The difference between the derivatives of order s $(0 \leq s \leq n)$ of two consecutive Bernstein polynomials, corresponding to a function f defined on the interval $[0, 1]$, can be expressed in the following form*

$$(B_{n+1}f)^{(s)}(x) - (B_n f)^{(s)}(x) =$$

$$= -\frac{1}{n(n+1)}\left[(n-1)^{[s]}x(1-x)\sum_{k=0}^{n-s-1} p_{n-s-1,k}(x)T_{n,k}^{(s+2)}(f)+\right.$$

$$+s(n-1)^{[s-1]}(1-2x)\sum_{k=0}^{n-s} p_{n-s,k}(x)T_{n,k}^{(s+1)}(f)-$$

$$\left.-s(s-1)\sum_{k=0}^{n-s-1} p_{n-s+1,k}(x)T_{n,k}^{(s)}(f)\right].$$

Examining the cases when the right side of the above equality has a constant sign, we can state

Theorem 3.2 (D.D. Stancu, [21], Th.4.2) *For the sequence of derivatives of order $s(s\geq 0)$ of the Bernstein polynomials $(B_nf)^{(s)}$, $n\in\mathbf{N}$, the following monotonicity properties hold:*

(i) For $s\geq 2$; if on the interval $[0,1/2]$ the function f is concave (convex) of order $s-1$ and convex (concave) of orders s and $s+1$, then the sequence is decreasing (increasing) on $[0,1/2]$; if on the interval $[1/2,1]$ the function f is concave (convex) of orders $s-1$ and s and convex (concave) of order $s+1$, then the sequence is decreasing (increasing) on $[1/2,1]$.

(ii) For $s=1$; if on the interval $[0,1/2]$ the function f is convex (concave) of first and second order, then the sequence is decreasing (increasing) on this interval; if on the interval $[1/2,1]$ the function f is concave (convex) of first order and convex (concave) of second order, then the sequence is decreasing (increasing) on this interval.

Of course, Theorem 3.1 allows us to study the case $s=0$ too.

3. The n^{th} Meyer-König and Zeller operator in a slightly modified version due to E.W. Cheney and A. Sharma is defined by

$$(M_nf)(x)=\sum_{k=0}^{\infty} m_{n,k}(x)f\left(\frac{k}{n+k}\right),\quad 0\leq x<1,$$

where $m_{n,k}(x)=(1-x)^{n+1}x^k\binom{n+k}{k}$ and $f\in C([0,1])$.

It maps the space $C([0,1])$ into the space $C([0,1))$. $(M_nf)(1)$ is defined as $(M_nf)(1)=\lim (M_nf)(x)=f(1)$. In 1970 A. Lupas and M.W. Müller [13] have revealed many properties of this sequence among them being its monotonicity properties.

Theorem 3.3 (Lupaş and Müller, [13], Th.3.3) *On the subspaces of $C([0,1])$ which are formed by convex, non-concave, polynomial, non-convex resp. concave functions of the first order on $[0,1]$, the sequence of the Meyer-König and Zeller operators is decreasing, non-increasing, stationary, non-decreasing resp. increasing.*

This result is based on the following identity

$$(M_{n+1} - M_n)(f) = -\sum_{k=0}^{\infty} m_{n,k}\mu_{k,n}(f)$$

where

$$\mu_{k,n}(f) =$$

$$= \frac{1}{(k+n+1)(k+n+2)} \left[\frac{k}{k+n+1}, \frac{k+1}{k+n+2}, \frac{k+1}{k+n+1}; f\right].$$

A. Lupaş [12] has also established the following property: the operator M_n, $n \in \mathbf{N}$, maps convex functions of $-1, 0, 1$ resp. 2 order into functions which are convex of the same order.

4. In 1980 Bleimann, Butzer and Hahn introduced a sequence of positive linear operators L_n defined on the the space $\mathbf{R}^{[0,\infty)}$ by

$$(L_n f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n+1-k}\right), \quad x \geq 0.$$

In the last two decades, this sequence was the object of an intensive study developed by many authors. To the best of our knowledge, the most recent investigation and well-done as well, is due to U. Abel and M. Ivan [1]. It exists a close connection between L_n and Bernstein operator B_n which was pointed out both by M. Ivan in [9] and J.A. Adell, F.G. Badía, J. de la Cal in [2]. Indeed, introducing the operators $U : \mathbf{R}^{[0,\infty)} \to \mathbf{R}^{[0,1]}$,

$V : \mathbf{R}^{[0,1]} \to \mathbf{R}^{[0,\infty)}$, defined respectively by

$$(Uf)(t) = \begin{cases} (1-t)f\left(\dfrac{t}{1-t}\right), & t \in [0,1) \\ 0, & t = 1 \end{cases}$$

$$(Vg)(x) = (1+x)g\left(\frac{x}{1+x}\right), \quad x \geq 0,$$

one obtains $L_n = V \circ B_{n+1} \circ U$, $n \in \mathbf{N}$. Consequently, many properties of L_n can be obtained easily from the corresponding properties of B_n.

By direct calculation, B. Della Vecchia has obtained [6], Eq. (2.4) the following formula

$$(L_n f)(x) - (L_{n+1}f)(x) = -[n, n+1; f]\left(\frac{x}{1+x}\right)^{n+1} +$$

$$+\frac{1}{(1+x)^{n+1}}\sum_{k=0}^{n-1}\frac{\binom{n+1}{k}x^{k+1}}{(n-k)(n+1-k)}\times$$

$$\times\left[\frac{k}{n+1-k}, \frac{k+1}{n+1-k}, \frac{k+1}{n-k}; f\right].$$

A consequence of this identity is the following result

Theorem 3.4 (see [6] or [2], Th.6) *If $f \in \mathbf{R}^{[0,\infty)}$ is convex and non-increasing or if $f \in C([0,\infty))$ is convex and $f(x) = o(x)$ $(x \to \infty)$, then $L_n f$ is convex and $f \leq L_{n+1}f \leq L_n f$, $n \in \mathbf{N}_0$.*

Della Vecchia [6] and R.A. Khan [10] have studied the preservation of convexity of order $r \leq 1$ for L_n. The general problem for an arbitrary $r \geq 0$ was solved in [2].

Theorem 3.5 (U. Abel, M. Ivan [2], Th.6.1) *Let $r \geq 0$, $n \in \mathrm{N}$ and $f \in \mathbf{R}^{[0,\infty)}$. We suppose that, for each $m = \overline{1, r+1}$ and all $\nu = \overline{0, n-1}$, we have*

$$(-1)^{r+m+1}[x_{n,\nu}, \ldots, x_{n,\nu+m}; w_{m-1}f] \geq 0$$

resp.,

$$(-1)^{r+m+1}[x_{n,\nu},\dots,x_{n,\nu+m};w_{m-1}f]\le 0,$$

where $x_{n,k}=k/(n+1-k)$ *and* $w_m(x)=(1+x)^m$, $x\ge 0$.

Then the Bleimann, Butzer and Hahn operator maps f *onto a function* L_nf *which is convex, resp., concave of order* r *on* $[0,\infty)$.

5. Let S_n be the Favard-Szasz-Mirakyan operator defined by

$$(S_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\quad x\ge 0,$$

where $f\in E_\infty$, $E_\infty:=\bigcup_{\alpha>0}E_\alpha$ and

$$E_\alpha:=\left\{f\in C([0,\infty)):\ \sup_{x\ge 0}e^{-\alpha x}|f(x)|<\infty\right\}.$$

The space E_α, $\alpha>0$, endowed with the natural order and the norm $\|\cdot\|_\alpha$, $\|f\|_\alpha=\sup_{x\ge 0}e^{-\alpha x}|f(x)|$ becomes a Banach lattice.

It has already been established [20], p.1191, that the following relation between two consecutive terms of the sequence $(S_nf)_{n\ge 1}$ holds

$$(S_{n+1}f)(x)-(S_nf)(x)=$$
$$=-\frac{x}{n(n+1)}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right],\quad x\ge 0.$$

By using the functionals $T_{n,k}^{(\nu)}$ defined by (3.1) where we choose $a_{k+i}=(k+i)/(n+i)$, $b_k=k/n$, $k\in\mathbf{N}_0$, $i\in\{0,1\}$, it results

$$(S_{n+1}f)(x)-(S_nf)(x)=-\frac{x}{n(n+1)}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}T_{n,k}^{(2)}f.\tag{3.2}$$

B. Della Vecchia [4] has introduced the operator ${}^rS_n$ defined by

$$({}^rS_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{x^k}{k!}n^{k+r-1}f\left(\frac{k}{n}\right)$$

for every $r \in \mathbf{N}$ and $f \in E_\infty$. Consequently from (3.2) we get

$$(S_{n+1}f)(x) - (S_nf)(x) = -\frac{x}{n(n+1)} \ {}^1S_n(T^{(2)}_{n,k}f;x)$$

and we obtain the following theorem:

Theorem 3.6 (Della Vecchia, [4], Th.3.1) *The following relationship holds*

$$(S_{n+1}f - S_nf)^{(m)}(x) = -\frac{1}{n(n+1)}(x \ {}^{m+1}S_n(T^{(m+2)}_{n,k}f;x) +$$

$$+m \ {}^mS_n(T^{(m+1)}_{n,k}f;x)), \quad m \in \mathbf{N}.$$

This way the monotonicity of the derivatives of the sequences $(S_nf)^{(m)}_{n\geq 1}$ was revealed.

Theorem 3.7 (Della Vecchia, [4], Cor.3.2) *For the sequence of derivatives of order m ($m \geq 0$) of the Szasz-Mirakyan operators $(S_nf)^{(m)}$, $n \in \mathbf{N}$, the following monotonicity property holds: if on the interval $[0,\infty)$ the function f is convex (concave) of order m and $m+1$, then the sequence is decreasing (increasing) in $[0,\infty)$.*

For the particular case $m = 1$ the above result was established earlier by I. Horová in 1982, [8].

6. Let V_n be the n^{th} Baskakov operator i.e.

$$(V_nf)(x) = \frac{1}{(1+x)^n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \quad x \geq 0,$$

where $f \in W_\infty$, $W_\infty := \bigcup_{m\geq 2} W_m$ and

$$W_m := \left\{f \in C([0,\infty)) : \lim_{x\to\infty}\frac{f(x)}{1+x^m} = 0\right\}.$$

The space W_m, for any integer $m \geq 2$, endowed with the natural order and the norm $\|\cdot\|_m$, $\|f\|_m = \sup_{x\geq 0} w_m(x)|f(x)|$,

$w_m(x) = (1 + x^m)^{-1}$, becomes a Banach lattice. Regarding these spaces and the spaces introduced in the previous paragraph we claim $W_\infty \subset E_\alpha \subset E_\infty$ for each $\alpha > 0$.

G. Mastroianni [14] has proved

$$(V_{n+1}f)(x) - (V_n f)(x) =$$

$$= -\frac{x}{n(n+1)} \sum_{k=0}^{\infty} \frac{(n+k+1)\dots(n+2)}{(1+x)^{n+k+1}} \frac{x^k}{k!} \left[\frac{k}{n+1}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right],$$

$x \geq 0$.

Following the same line presented by us in the previous paragraph, Della Vecchia [4] has introduced for any $r \in \mathbf{N}$ and $f \in W_\infty$ the linear positive operator ${}^rV_n$ as

$$({}^rV_n f)(x) = \sum_{k=0}^{\infty} \frac{(n+k+1)\dots(n+2)}{(1+x)^{n+k+r}} \frac{x^k}{k!} f\left(\frac{k}{n}\right).$$

By making use of the functionals $T_{n,k}^{(\nu)}$ from (3.1) and choosing $a_k = k/(n+1)$, $b_k = k/n$, $k \in \mathbf{N}_0$, one obtains

$$(V_{n+1}f)(x) - (V_n f)(x) = -\frac{x}{n(n+1)} \; {}^1V_n(T_{n,k}^{(2)} f; x).$$

Starting by this identity one has

Theorem 3.8 (Della Vecchia, [4], Th.4.1) *The following relationship holds:*

$$(V_{n+1}f - V_n f)^{(m)}(x) =$$

$$= -\frac{m!}{n(n+1)} \Bigg\{ x \Bigg({}^{m+1}V_n \Bigg(\binom{n+k+m}{m} T_{n,k}^{(m+2)} f +$$

$$+ \binom{n+k+m}{m-1} T_{n,k+1}^{(m+1)} f; x \Bigg) \Bigg) +$$

$$+ {}^mV_n \Bigg(\binom{n+k+m-1}{m-1} T_{n,k}^{(m+1)} f +$$

$$+\binom{n+k+m-1}{m-2}T^{(m)}_{n,k+1}f;x\Big)\Big\}, \quad m \in \mathbf{N}.$$

Taking into account this result, we have

Theorem 3.9 (Della Vecchia, [4], Cor. 4.2) *For the sequence $(V_n f)^{(m)}$, $n \in \mathbf{N}$, the following monotonicity properties hold:*

(i) For $m = 1$; if on the interval $[0, \infty)$ the function f is convex (concave) of first and second order, then the sequence is decreasing (increasing) in $[0, \infty)$.

(ii) For $m \geq 2$; if on the interval $[0, \infty)$ the function f is convex (concave) of order $m-1$, m and $m+1$, then the sequence is decreasing (increasing) in $[0, \infty)$.

The particular case $m = 0$ was investigated in 1966 by A. Lupaş [11].

Remark 3.1 *Regarding the operators S_n, V_n for any bounded function f belonging to the space $\mathbf{R}^{[0,\infty)}$, Della Vecchia has established [5], Ths.2.1 and 2.2*

(i) If f is non-concave (non-convex) of order m and $m+1$ on $[0, \infty)$ then

$$[x_0, x_1, \ldots, x_m; S_{n+1}f] \leq (\geq)[x_0, x_1, \ldots, x_m; S_n f]$$

holds for any $x_0, x_1, \ldots, x_m$ non-negative numbers and $(n, m) \in \mathbf{N} \times \mathbf{N}$.

(ii) If f is non-concave (non-convex) of order $m-1, m$ and $m+1$ on $[0, \infty)$ then

$$[x_0, x_1, \ldots, x_m; V_{n+1}f] \leq (\geq)[x_0, x_1, \ldots, x_m; V_n f]$$

holds, for any $x_0, x_1, \ldots, x_m$ non-negative numbers and $(n, m) \in \mathbf{N} \times \mathbf{N}$.

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