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BINOMIAL POLYNOMIALS AND THEIR APPLICATIONS IN APPROXIMATION THEORY

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Binomial polynomials and their applications in approximation theory

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Abstract

In this paper we are concerned with the sequences of polynomials of binomial type. In particular we point out their remarkable algebraic–combinatorial properties related to the so called delta operators as used in a series of papers on the foundations of combinatorial theory, see [23], [27]. In order to detail this field, the theoretical aspects are illustrated with several concrete examples. The paper is also a survey of the role of these polynomials in Approximation Theory and it includes the construction of general binomial type operators and their main approximation properties.

Introduction

The main objective of this survey paper is to present the role of binomial polynomials in Approximation Theory, more precisely, a unified theory of the approximation operators of binomial type by exploiting the technique of the *umbral calculus* or *symbolic calculus*, widely used in the past century. In its modern form, this is a powerful tool for calculations with polynomials. It has its origin in so–called Heaviside calculus created by G. Boole and extended by A. Cayley, P. Appell, S. Pincherle, J. Blissard and after 1900 by N. Nielsen, N.E. Nörlund, J.M. Sheffer, E.T. Bell. We point out that the first rigorous version of this calculus belongs to Gian–Carlo Rota and his collaborators; among them we mention R. Mullin, S. Roman, D. Kahaner, A. Odlyzko. The umbral calculus is a successful combination between the finite differences calculus and certain chapters of Functional Analysis and Probability Theory.

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The paper is organized in three sections:
1. Notation and preliminaries,
2. Delta operators and their basic polynomials,
3. Approximation operators of binomial type.

Firstly we introduce the notion of polynomial sequence of binomial type and some basic facts needed in the subsequent analysis accompanied by several examples.

The focus of the second section is to present delta operators, the Pincherle derivative of a linear operator and Sheffer sequences. We investigate their main properties and the connections between them. We establish both identities involving the generating function of a binomial sequence and explicit formulas for the basic polynomials associated to a given delta operator. As an illustration of the theory a number of examples are also given.

The third section contains the application of the binomial sequences in the construction of linear approximation processes. For the analysed operators we give quantitative estimates of the rate of convergence. We also show that such operators leave invariant the cone of the convex functions of higher order and preserve the Lipschitz constants. In particular cases some classical operators are reobtained. Further, we shall examine some integral extensions of these operators. The final part in devoted to the link between exponential–type operators and the basic set of polynomials of binomial type.

1. Notation and preliminaries

Throughout the paper the symbol $\mathbb{N}_0$ stands for the set $\mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}_0$ we denote by Π_n the linear space of polynomials of degree no greater than n and by Π_n^* the set of all polynomials of degree n. We set

$$\Pi := \bigcup_{n\geq 0} \Pi_n.$$

Π represents the commutative algebra of polynomials with coefficients in $\mathbb{K}$, this symbol standing either for the field $\mathbb{R}$ or for the field $\mathbb{C}$.

A sequence $p = (p_n)_{n\geq 0}$ such that $p_n \in \Pi_n^*$ for every $n \in \mathbb{N}_0$ will be called a *polynomial sequence.*

Definition 1.1. *A polynomial sequence $b = (b_n)_{n\geq 0}$ is called of binomial type if for any $(x, y) \in \mathbb{K} \times \mathbb{K}$ the following equalities*

$$b_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), \quad n \in \mathbb{N}_0, \tag{1.1}$$

hold.

Remark 1.2. Knowing that $deg(b_0) = 0$ we get $b_0(x) = 1$ for any $x \in \mathbb{K}$ and by induction we easily obtain $b_n(0) = 0$ for any $n \in \mathbb{N}$.

Examples 1.3. The most common example of binomial sequence is $e = (e_n)_{n\geq 0}$, $e_n(x) = x^n$ (the monomials). We will keep this notation throughout the paper. Some nontrivial examples are given below.

1.3.1. The **generalized factorial power with the step** a : $p = (p_n)_{n\geq 0}$, $p_0(x) = x^{[0,a]} := 1$ and $p_n(x) = x^{[n,a]} := x(x-a)\ldots(x-(n-1)a)$, $n \in \mathbb{N}$.

The Vandermonde formula, i.e. $(x+y)^{[n,a]} = \sum_{k=0}^{n} \binom{n}{k} x^{[k,a]} y^{[n-k,a]}$, guarantees that this is a binomial type sequence. There are two particular cases: for $a = 1$ we obtain the **lower–factorials** which, usually, are denoted by $\langle x\rangle_n$; for $a = -1$ we obtain the **upper–factorials** denoted by Pochhammer's symbol $(x)_n$. It clearly appears that $(x)_n = \langle x+n-1\rangle_n$. We also we recall that $x^{[-n,a]} := 1/(x+na)^{[n,a]}$.

1.3.2. The **exponential polynomials** introduced by J.F. Steffensen and studied by Touchard: $t = (t_n)_{n\geq 0}$, $t_n(x) = \sum_{k=0}^{n} S(n,k)x^k$ where $S(n,k)$, $k \in \mathbb{N}_0$, represent the Stirling numbers of the second kind defined by $x^n = \sum_{k=0}^{n} S(n,k)\langle x\rangle_k$, or explicitely, by using the divided differences, $S(n,k) = [0,1,\ldots,k;e_n]$. We recall that the following identity is known in literature [23] as Dobinski formula

$$t_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k.$$

1.3.3. **Abel polynomials**: $\tilde{a} = (a_n^{\langle a\rangle})_{n\geq 0}$, $a_0^{\langle a\rangle} = 1$, $a_n^{\langle a\rangle}(x) = x(x-na)^{n-1}$, $n \in \mathbb{N}$, $a \neq 0$. Rewriting the identity (1.1) for these polynomials we obtain the Abel–Jensen combinatorial formula

$$(x+y)(x+y+na)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} xy(x+ka)^{k-1}(y+(n-k)a)^{n-1-k}, \quad n \in \mathbb{N}.$$

1.3.4. **Gould polynomials**: $g = (g_n^{\langle a,b\rangle})_{n\geq 0}$, $g_0^{\langle a,b\rangle} = 1$, $g_n^{\langle a,b\rangle}(x) = \dfrac{x}{x-an}\left\langle \dfrac{x-an}{b}\right\rangle_n$, $n \in \mathbb{N}$, $ab \neq 0$.

The space of all linear operators $T : \Pi \to \Pi$ will be denoted by $\mathcal{L}$. Among these operators an important role will be played by the shift operators, named E^a. For every $a \in \mathbb{K}$, E^a is defined by $(E^a p)(x) = p(x+a)$, where $p \in \Pi$. An operator $T \in \mathcal{L}$

which commuted with all shift operators, that is $TE^a = E^aT$ for every $a \in \mathbb{K}$, is called a *shift–invariant operator* and the set of these operators will be denoted by $\mathcal{L}_s$.

2. Delta operators and their basic polynomials

2.1. Delta operators

A special class of $\mathcal{L}_s$ is formed by the so called *theta operators*, a term introduced by J.F. Steffensen [34] in 1927. In 1956 F.B. Hildebrand [11] called them *delta operators* and this term was taken over and intensively used by Gian–Carlo Rota and his collaborators. Therefore we will continue to use this latest term.

Definition 2.1. *An operator Q is called a delta operator if $Q \in \mathcal{L}_s$ and Qe_1 is a nonzero constant.*

Let $\mathcal{L}_\delta$ denote the set of all delta operators.

Examples 2.2. Here we present some examples of delta operators. The symbol I stands for the identity operator on the space Π.

2.2.1. The derivative operator, denoted by D.

2.2.2. The operators used in calculus of divided differences. Let h be a fixed number belonging to the field $\mathbb{K}$. We set $\Delta_h := E^h - I$, **the forward difference operator**; $\nabla_h := I - E^{-h}$, **the backward difference operator**; $\delta_h := E^{h/2} - E^{-h/2}$, **the central difference operator**; $M_h := (1/2)(I + E^h)$, **the mean value operator**. It is evident that $\nabla_h = \Delta_h E^{-h}$, $\delta_h = \Delta_h E^{-h/2} = \nabla_h E^{h/2}$, $M_h = I + (1/2)\Delta_h = E^h - (1/2)\Delta_h$. The properties of these operators can be found in [14, Chapter 1].

2.2.3. **Abel operator**, $A_a := DE^a$. For any $p \in \Pi$, $(A_a p)(x) = \dfrac{dp}{dx}(x+a)$. Writing (symbolically) the Taylor's series in the following manner

$$E^h = \sum_{\nu=0}^{\infty} \frac{h^\nu D^\nu}{\nu!} = e^{hD}, \tag{2.1}$$

we can also get $A_a = D(e^{aD})$.

2.2.4. **Gould operator**, $G_{a,b} := \Delta_b E^a = E^{a+b} - E^a$, $ab \neq 0$.

2.2.5. **Laguerre operator**, $L := \dfrac{D}{D-I}$. More clearly, $L = -D(I-D)^{-1} = -D - D^2 - D^3 - \ldots$ Notice that there are no convergence problems with this power series in D since the infinite sum reduces to a finite one when applied to any polynomial p. Also we recall that

$$(Lp)(x) = -\int_0^\infty \frac{dp}{dt}(x+t)dt.$$

2.2.6. **Touchard operator**, $T := \log(I + D) = D - \frac{D^2}{2} + \frac{D^3}{3} - \dots$, for any polynomial the sum being also finite. Another representation of this operator is the following one:

$$(Tp)(x) = \int_0^\infty e^{-t}[x, x-t; p]dt.$$

Actually, delta operators possess some of the properties of the derivative operator. For example,

$$if\ Q \in \mathcal{L}_\delta\ then\ Qa = 0\ for\ every\ constant\ a.$$

Indeed, because Q is a delta operator we have $Qe_1 = c \neq 0$ as well as $Q \in \mathcal{L}_s$, consequently $(QE^a)(e_1) = (E^aQ)(e_1)$. The left member of this identity can be written as $Q(e_1 + a) = c + Qa$, the right member is $E^a c = c$ and hence we obtain $Qa = 0$.

More generally, according to [27, Proposition 2] for every $Q \in \mathcal{L}_\delta$ we have

$$Q(\Pi_n^*) \subset \Pi_{n-1}^*, \quad n \in \mathbb{N}.$$

Definition 2.3. *Let Q be a delta operator. A polynomial sequence $p \in (p_n)_{n\geq 0}$ is called the sequence of basic polynomials associated to Q if*

(i) $p_0(x) = 1$ *for any* $x \in \mathbb{K}$,
(ii) $p_n(0) = 0$ *for any* $n \in \mathbb{N}$,
(iii) $(Qp_n)(x) = np_{n-1}(x)$ *for any* $n \in \mathbb{N}$ *and* $x \in \mathbb{K}$.

We mention that this term was used both by I.M. Sheffer [31] and by Gian–Carlo Rota [27] and his collaborators. The above polynomials p_n, $n \in \mathbb{N}_0$, were called by J.F. Steffensen [34] *poweroids*, considering that they represent an extension of the notion of power.

If $p = (p_n)_{n\geq 0}$ is a sequence of basic polynomials associated to Q then $\{p_0, p_1, \dots, p_{n-1}, e_n\}$ is a basis of the vectorial space Π_n. Taking this fact into account by induction it can be proved [27, Proposition 3] that

every delta operator has a unique sequence of basic polynomials.

Examples 2.4.

We give some examples: $(e_n)_{n\geq 0}$, $(x^{[n,h]})_{n\geq 0}$ respectively $((x + (n-1)h)^{[n,h]})_{n\geq 0}$ represent the sequence of basic polynomials associated to $Q = D$, $Q = \Delta_h$ respectively $Q = \nabla_h$. Also, we can easily prove that $\tilde{a} = (a_n^{\langle a \rangle})_{n\geq 0}$ respectively $g = (g_n^{\langle a,b \rangle})_{n\geq 0}$ is the sequence of basic polynomials associated to Abel operator A_a respectively Gould operator $G_{a,b}$.

The so called Steffensen polynomial sequence $(p_n)_{n\geq 0}$, $p_0 = 1$, $p_1 = e_1$, $p_n(x) = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\dots\left(x + \frac{n}{2} - n + 2\right)$, $n \geq 2$, represents the basic set associated to $Q = \delta_1$ (the central difference operator with $h = 1$), see [26, page 115].

The following result establishes the connection between delta operator and the binomial type sequences. For its proof see [27, Theorem 1].

Theorem 2.5. (a) *If* $p = (p_n)_{n\geq 0}$ *is a basic sequence for some delta operator* Q, *then it is a sequence of binomial type.*

(b) *If* $p = (p_n)_{n\geq 0}$ *is a sequence of binomial type, then it is a basic sequence for some delta operator.*

Iterating the third property of the definition of basic polynomials we obtain $(Q^k p_n)(x) = \langle n\rangle_k p_{n-k}(x)$. Hence, for $k = n$ we have $(Q^n p_n)(0) = n!$, while for $k < n$, $(Q^k p_n)(0) = 0$ holds. Since any polynomial q is a linear combination of the above basic polynomials, we obtain

$$q(x) = \sum_{k=0}^{deg(q)} \frac{(Q^k q)(0)}{k!} p_k(x),$$

and consequently, by choosing $q := E^y q$, we get

$$q(x+y) = \sum_{k=0}^{deg(q)} \frac{(Q^k q)(y)}{k!} p_k(x).$$

This identity is the basic starting point which allows us to obtain the expansion of a shift–invariant operator in terms of a delta operator and its powers. The following theorem, named "first expansion theorem" [27, page 691] generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 2.6. *Let* T *be a shift–invariant operator and let* Q *be a delta operator with its basic sequence* $(p_n)_{n\geq 0}$. *Then the following identity*

$$T = \sum_{k\geq 0} \frac{(Tp_k)(0)}{k!} Q^k \tag{2.2}$$

holds.

Further, we point out the following algebraic result. Let Q be a delta operator and let $(\mathcal{F}, +, \cdot)$ be the ring of the formal power series in the variable t over the same field. Here the product means the Cauchy product between two series.

Let $(\mathcal{L}_s, +, \cdot)$ be the ring of shift–invariant operators. Here the product is defined as usually (for any $P_1, P_2 \in \mathcal{L}_s$ we have $P_1P_2 : \Pi \to \Pi$, $(P_1P_2)(q) = P_1(P_2(q))$ for every $q \in \Pi$). Then there exists an isomorphism ψ from $\mathcal{F}$ onto $\mathcal{L}_s$ such that

$$\psi(f(t)) = T \quad \text{where} \quad f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \quad \text{and} \quad T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k. \tag{2.3}$$

The mapping is clearly linear and by the first expansion theorem, it is onto. Therefore, we must only verify that the map preserves products — for this see [27, page 692].

Remarks 2.7. This isomorphism allows us to conclude:

(i) A shift–invariant operator T is invertible if and only if $Te_0 \neq 0$. Since for every $Q \in \mathcal{L}_\delta$ we have $Qe_0 = 0$ we see that no delta operator is invertible.

(ii) Since $(\mathcal{F}, +, \cdot)$ is a commutative ring, any two shift–invariant operators commute, that is $TS = ST$ for every $T, S \in \mathcal{L}_s$.

As a special case of Theorem 2.6. it follows that any shift–invariant operator T can be expressed in terms of D, that is

$$T = \sum_{k\geq 0} \frac{a_k}{k!} D^k, \quad \text{where} \quad a_k = (Te_k)(0). \tag{2.4}$$

Further, by the isomorphism (2.3), the formal power series corresponding to T is $f_T(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k$. We call $f_T(t)$ *the indicator of* T.

Taking into account Examples 2.2. we present the indicators of some operators: $f_D(t) = t$, $f_{\Delta_h}(t) = e^{ht} - 1$, $f_{\nabla_h}(t) = 1 - e^{-ht}$, $f_{A_a}(t) = te^{at}$, $f_G(t) = e^{at}(e^{bt} - 1)$, $f_L(t) = t(t-1)^{-1}$, $f_T(t) = \log(1+t)$.

In the following we shall write $T = \phi(D)$, where $T \in \mathcal{L}_s$ and $\phi(t)$ is a formal power series, to indicate that the operator T corresponds to the series $\phi(t)$ under the isomorphism defined by (2.3).

It is known that to every series $y = \sum_{i\geq 1} c_i x^i$, $c_1 \neq 0$, corresponds a unique inverse power series $x = \sum_{i\geq 1} C_i y^i$, where $C_1 = c_1^{-1}$, $C_2 = -c_2 c_1^{-3}$, $C_3 = (2c_2^2 - c_1 c_3)c_1^{-5}$, $C_4 = (5c_1c_2c_3 - c_1^2 c_4 - 5c_2^3)c_1^{-7}$, and so on, for details see [1; *3.6.25.* Reversion of Series]. For our formal power series $\phi(t)$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$ we denote by $\phi^{-1}(t)$ its inverse series; consequently

$$\text{if} \quad \phi(t) = \sum_{k\geq 1} \frac{a_k}{k!} t^k \quad \text{then} \quad \phi(\phi^{-1}(t)) = \sum_{k\geq 1} \frac{a_k}{k!} (\phi^{-1}(t))^k = t = \phi^{-1}(\phi(t)),$$

where the sum is well defined, since $\phi^{-1}(0) = 0$ and $(\phi^{-1})'(0) \neq 0$.

Now we can give a characterization of any delta operator.

Theorem 2.8. *An operator $P \in \mathcal{L}_s$ is a delta operator if and only if it corresponds, under the isomorphism defined by (2.2), to a formal power series $\phi(t)$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$.*

We are going to present the identity involving the generating function of a binomial sequence.

Firstly, we consider a given $Q \in \mathcal{L}_\delta$ and its basic sequence $(p_n)_{n\geq 0}$. By using (2.4) this operator can be written $Q = \phi(D)$. According to the previous results, let's define the formal power series $t = \phi(u)$ and $u = \phi^{-1}(t)$.

Secondly, in relation (2.2) we choose $T = E^x$ and expand E^x in terms of Q. Knowing both $(E^x p_k)(0) = p_k(x)$ and the identity (2.1) one gets

$$e^{xD} = \sum_{k\geq 0} \frac{p_k(x)}{k!} \phi^k(D).$$

Substituting D by u, the series terms lead us to the following result.

Theorem 2.9. ([27, Corollary 3]) *Let Q be a delta operator with $p = (p_n)_{n\geq 0}$ its sequence of basic polynomials. Let $\phi(D) = Q$ and $\varphi(t)$ be the inverse formal power series of $\phi(u)$. Then*

$$e^{x\varphi(t)} = \sum_{n\geq 0} \frac{p_n(x)}{n!} t^n, \tag{2.5}$$

where $\varphi(t)$ has the form $c_1 t + c_2 t^2 + \ldots$ $(c_1 \neq 0)$.

At this moment, by using the properties of the exponential function and the definition of the Cauchy product we easily check that the sequence p satisfies condition (1.1) thus it is of binomial type. The map $t \mapsto e^{\varphi(t)}$ implies the existence of a series $\theta(t) = 1 + d_1 t + d_2 t^2 + \ldots$ such that $[\phi(t)]^x = e^{x\varphi(t)}$.

According to (2.5) it is also clear that one can obtain sequences of binomial type by using *the generating functions method.*

We present another characterization of delta operators which appears in [27, Proposition 4]. Because there the result is only listed we include the proof.

Theorem 2.10. *$Q \in \mathcal{L}_s$ is a delta operator if and only if $Q = DP$ for some shift–invariant operator P, where the inverse operator P^{-1} exists.*

Proof. If in (2.4) we substitute T by a delta operator Q then we get $a_0 = Q(e_0) = 0$ and $a_1 = Q(e_1) = c \neq 0$. Consequently, we can write

$$Q = \sum_{k\geq 1} \frac{a_k}{k!} D^k. \tag{2.6}$$

Denoting $\sum_{k\geq 1} \frac{a_k}{k!} D^{k-1}$ by P we have $P \in \mathcal{L}_s$ and $P(e_0) = a_1 \neq 0$, thus P is invertible, see 2.7.(i). So, Q can be written as DP. Reciprocally, for every $P \in \mathcal{L}_s$ such that P is invertible, DP is a shift–invariant operator, $E^a(DP) = (DP)E^a$, and $(DP)(e_1) = P(D(e_1)) = P(e_0) = c \neq 0$ thus $DP \in \mathcal{L}_\delta$. □

2.2. Delta operators and Pincherle derivatives

The operator $X : \Pi \to \Pi$, $(Xp)(x) = xp(x)$, is called *the multiplication operator.* Clearly, X is a linear but not a shift–invariant operator.

Definition 2.11. *Let U belong to $\mathcal{L}$ and let X be the multiplication operator. The operator $U' := UX - XU$ is called the Pincherle derivative of U.*

For example, by using the above definition, we get $I' = 0$, $D' = I$, $(D^k)' = kD^{k-1}$, $(E^a)' = aE^a$.

We indicate some properties of the Pincherle derivative of $U \in \mathcal{L}$.

Theorem 2.12.

(i) *If U is a shift–invariant operator, then U' is also a shift–invariant operator.*
(ii) *The following formula $(UV)' = U'V + UV'$ holds for every $U, V \in \mathcal{L}$.*
(iii) *If $U \in \mathcal{L}_s$ has the indicator $f_U(t)$, then U' has $\frac{d}{dt} f_U(t)$ as its indicator.*
(iv) *If the n^{th} Pincherle derivative of U is defined by $U^{(n)} = (U^{(n-1)})'$, $n \in \mathbb{N}$, then the following identity*

$$U^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} X^i U X^{n-i},$$

holds, for every $U \in \mathcal{L}$.
(v) *If $U \in \mathcal{L}_\delta$, then U' is invertible.*

Proof. By using the definition of Pincherle derivative, a straightforward calculation leads us to the first two former statements. The third property is a consequence of (2.4), the definitions of $f_U(t)$ and U' as well as of the relation $(D^k)' = kD^{k-1}$. The fourth one can be proved by induction, see e.g. [21, Lemma 3.6]. The last statement follows from the previous property (iii) and the isomorphism relation (2.3). □

We are ready to present explicit formulas for the basic polynomials associated to a given delta operator. For the proof see [27; Theorem 4].

Theorem 2.13. *Let $Q \in \mathcal{L}_\delta$ and $R = Q'^{-1}$. Let $p = (p_n)_{n \geq 0}$ be the sequence of basic polynomials associated to Q. For every $n \in \mathbb{N}$, the following identities*

(i) $p_n = (Q'P^{-n-1})(e_n)$, (ii) $p_n = P^{-n}(e_n) - (P^{-n})'(e_{n-1})$,
(iii) $p_n = e_1 P^{-n}(e_{n-1})$, (iv) $p_n = e_1 R(e_{n-1})$, *(Rodrigues formula)*,

hold, where $P \in \mathcal{L}_s$ such that P^{-1} exists and is given as in Theorem 2.10. by $DP = Q$.

Example 2.14. The above formulas are useful to construct the basic sequence of a given delta operator. Since we have not yet indicated the basic sequence of Laguerre operator L, we will do it now. We have $L = D(D - I)^{-1}$ thus $P = (D - I)^{-1}$. Applying formula (iii) of Theorem 2.13., for every $n \in \mathbb{N}$, we can write

$$p_n = e_1 (D - I)^n (e_{n-1}) = e_1 \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{i} D^i (e_{n-1}),$$

$$p_n(x) = x\sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i}(n-1)(n-2)\dots(n-i)x^{n-1-i} =$$

$$= \sum_{k=1}^{n}(-1)^k\binom{n}{n-k}(n-1)(n-2)\dots kx^k = \sum_{k=1}^{n}\frac{n!}{k!}\binom{n-1}{k-1}(-x)^k.$$

Notice that $\frac{1}{n!}p_n$, $n \in \mathbb{N}$, represent the Laguerre polynomials of order zero, see [22, Remark 2.7.4, page 44] and the coefficients $\frac{n!}{k!}\binom{n-1}{k-1}$ are known as the *Lash numbers*.

2.3 Delta operators and Sheffer sequences

Next, we connect the binomial sequences with the so called sequences of type zero. This notion was introduced by I.M. Sheffer [31, page 594] in 1939. The conditions for a set of type zero were stated as follows: $s = (s_n)_{n\geq 0}$ is of type zero if $s_0 \neq 0$ and $J(s_n) = s_{n-1}$, $n \in \mathbb{N}$, where the operator $J : \Pi \to \Pi$ is given for every $q \in \Pi$ by the relation

$$J(q) = l_1q' + l_2q'' + l_3q''' + \dots, \quad l_1 \neq 0.$$

It is easy to verify that J is actually a delta operator. To this end we can examine the relation (2.6). In the present paper the definition of a sequence of type zero comes from [27, page 698] and it slightly differs from the original approach.

Definition 2.15. *Let Q be a delta operator. The polynomial sequence $s = (s_n)_{n\geq 0}$ is of type zero (or a Sheffer sequence) for the operator Q if*

$$s_0 \neq 0 \quad and \quad Q(s_n) = ns_{n-1}, \quad n \in \mathbb{N}. \tag{2.7}$$

We mention that in [35] it appears a purely algebraic definition for Sheffer sequences which avoids the notion of formal power series.

A Sheffer set for $Q \in \mathcal{L}_\delta$ is related to the set of basic polynomials of Q by the following result whose proof can be found in [27, Proposition 1, p.698].

Theorem 2.16. *Let $Q \in \mathcal{L}_\delta$ with its basic sequence $(p_n)_{n\geq 0}$. Then $s = (s_n)_{n\geq 0}$ is a Sheffer sequence relative to Q if and only if there exists an invertible shift–invariant operator S such that*

$$s_n = S^{-1}(p_n), \quad n \in \mathbb{N}_0. \tag{2.8}$$

With the help of this result we can find out the defining property for Sheffer polynomials.

Since $(p_n)_{n\geq 0}$ is of binomial type we have the identity (1.1). We apply the shift–invariant operator S^{-1} to both sides (where x is the variable) and use (2.8) to obtain

$$\sum_{k=0}^{n}\binom{n}{k}s_k(x)p_{n-k}(y) = (S^{-1}p_n)(x+y) = (S^{-1}(E^y p_n))(x) =$$

$$= ((E^y S^{-1})(p_n))(x) = (E^y s_n)(x) = s_n(x+y).$$

We can now state the following result.

Theorem 2.17. *Let $Q \in \mathcal{L}_\delta$ with its basic sequence $(p_n)_{n\geq 0}$. For a Sheffer sequence $s = (s_n)_{n\geq 0}$ relative to Q the following identity holds*

$$s_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}s_k(x)p_{n-k}(y),$$

for every $n \in \mathbb{N}_0$ and $(x,y) \in \mathbb{K}\times\mathbb{K}$.

Remark 2.18. If we choose in the above identity $x = 0$ then

$$s_n(y) = \sum_{k=0}^{n}\binom{n}{k}s_k(0)p_{n-k}(y),$$

in other words, the polynomials s_n, $n \in \mathbb{N}_0$, are completely determined by their constant terms $s_k(0)$, $k \in \mathbb{N}_0$, and the basic sequence of Q.

We recall that in the particular case when Q becomes the ordinary differential operator D, the Sheffer polynomials relative to D are called *Appell polynomials.* These polynomials were introduced in 1880 by P. Appell [4]. The members of an Appell set $(A_n)_{n\geq 0}$ have the form $A_n(x) = \sum_{k=0}^{n} A_k(0)\binom{n}{k}x^{n-k}$, $x \in \mathbb{K}$, $n \in \mathbb{N}_0$.

A detailed study upon the Appell polynomials was carried out by Corrado Scaravelli, see [29], [30]. We can characterize a sequence of Appell polynomials by the following

Theorem 2.19. (B.C. Carlson, [6, Eq.(1.4)]) *The polynomials sequence $(A_n)_{n\geq 0}$ is of Appell type if and only if a generating function exists having the form*

$$e^{xt}g(t) = \sum_{n\geq 0} A_n(x)\frac{t^n}{n!}, \tag{2.9}$$

where $g(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!}$, $a_0 \neq 0$.

For example, choosing in (2.9) $g(t) = t/(e^t - 1)$, $|t| < 2\pi$, we obtain Bernoulli polynomials and choosing $g(t) = 2/(e^t + 1)$, $|t| < \pi$, we obtain Euler polynomials, see [1, *23.1.* page 804] or [10, Vol.3, page 253].

3. Approximation operators of binomial type

Our aim is to present an application of the binomial sequences, concerning the construction of sequences of approximation linear operators.

3.1 Operators of binomial type

We consider a delta operator Q and its sequence of basic polynomials $p = (p_n)_{n\geq 0}$, under the assumption that $p_n(1) \neq 0$ for every $n \in \mathbb{N}$. Also, according to Theorem 2.9. we shall keep the same meaning of the functions ϕ and φ. For every $n \geq 1$ we consider the linear operators $L_n^Q : C([0,1]) \to C([0,1])$ defined as follows:

$$(L_n^Q f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \tag{3.1}$$

They are called (cf. e.g., P. Sablonniere [28]) *Bernstein–Sheffer operators*, but as D.D. Stancu and M.R. Occorsio motivated in [33], these operators can be named *Popoviciu operators*. In 1931 Tiberiu Popoviciu [24] indicated the construction (3.1), in front of the sum appearing the factor d_n^{-1} from the identities $(1 + d_1 t + d_2 t^2 + \ldots)^x = e^{x\varphi(t)} = \sum_{n=0}^{\infty} p_n(x) t^n / n!$, see (2.5). If we choose $x = 1$ it becomes obvious that $d_n = p_n(1)/n!$.

In the particular case $Q = D$, L_n^D becomes the Bernstein operator of degree n.

The operators L_n^Q, $n \in \mathbb{N}$, are linear and reproduce the constants. Indeed, choosing in (1.1) $y := 1 - x$ we obtain $L_n^Q e_0 = e_0$.

The positivity of these operators are given by the sign of the coefficients of the series $\varphi(t) = c_1 + c_2 t + \ldots$ $(c_1 \neq 0)$. More precisely, T. Popoviciu [24] and later P. Sablonniere [28, Theorem 1] have established that

Lemma 3.1. *L_n^Q is a positive operator on $C([0,1])$ for every $n \geq 1$ if and only if $c_1 > 0$ and $c_n \geq 0$ for all $n \geq 2$.*

The next theorem collects the most significant results concerning these operators.

[Sablonnière, Th2(i)]

Theorem 3.2. *If the operator L_n^Q defined by (3.1) satisfies the conditions of Lemma 3.1. then the following statements are true.*

(i) *L_n^Q is an isomorphism of Π_n preeserving the degree, i.e., $L_n^Q q \in \Pi_k$ whenever $q \in \Pi_k$, $0 \le k \le n$.*

(ii) *One has $L_n^Q e_j = e_j$, $j \in \{0,1\}$ for any $n \in \mathbb{N}$ and $L_n^Q e_2 = e_2 + a_n(e_1 - e_2)$, for any $n \ge 2$, where $a_n = \frac{1}{n}\left(1 + (n-1)\frac{r_{n-2}(1)}{p_n(1)}\right)$, the sequence $(r_n(x))_{n\ge 0}$ being generated by*

$$\varphi''(t)\exp(x\varphi(t)) = \sum_{n\ge 0} r_n(x)\frac{t^n}{n!}. \tag{3.2}$$

(iii) *$L_n^Q f$ converges uniformly to $f \in C([0,1])$ if and only if the condition $\lim_{n\to\infty}(r_{n-2}(1)/p_n(1)) = 0$ holds.*

(iv) *If $r_{n-2}(1)/p_n(1) = \mathcal{O}(n^{-1})$ then there exists an integer $k \ge 1$ for which $\varphi \in \Pi_k$ and we have $\|L_n^Q f - f\|_\infty \le \left(1 + \sqrt{k}/2\right)\omega_1(f; 1/\sqrt{n})$.*

Here $\|\cdot\|_\infty$ is the sup–norm of the Banach space $C([0,1])$ and $\omega_1(f;\cdot)$ is the first modulus of continuity of f.

(v) *If $f \in Lip_M\alpha$ then $L_n^Q f \in Lip_M\alpha$ where*

$$Lip_M\alpha := \{f \in C([0,1]) : \ \omega_1(f;t) \le Mt^\alpha, \ 0 < t \le 1\}, \quad 0 < \alpha \le 1.$$

Remarks 3.3. (i) Statement (iv) shows that the classical Bernstein operators ($k = 1$) could be considered as the best positive Bernstein–Sheffer–Popoviciu operators associated with functions φ which are polynomials.

(ii) Before Sablonniere's paper, in 1984 C. Manole [19, pages 97–98], by using the Pincherle derivative, gave a form of $L_n^Q e_2$ as follows

$$L_n^Q e_2 = e_2 + \left(\frac{1}{n} + a_n^{(2)}\right)(e_1 - e_2), \text{ where } a_n^{(2)} = \frac{n-1}{n}\left(1 - \frac{(Q'^{-2}p_{n-2})(1)}{p_n(1)}\right), \ n \ge 2. \tag{3.3}$$

Accordingly, the condition of the third statement can be substituted by

$$\lim_{n\to\infty}\frac{(Q'^{-2}p_{n-2})(1)}{p_n(1)} = 1.$$

(iii) Let $Lip_M^*\alpha = \left\{f \in C([0,1]) : \ \omega_2(f;h) \le Mh^\alpha, \ 0 < h \le \frac{1}{2}\right\}$ be Lipschitz classes with respect to the second order modulus of continuity. In [8] C. Cottin and H.H. Gonska proved that

$$f \in Lip_M^*\alpha \quad \text{implies} \quad B_n f \in Lip_{4,5M}^*\alpha, \quad 0 < \alpha \le 2,$$

$B_n = L_n^D$ being the Bernstein operator. Motivated by the result of T. Lindvall [15] the authors conjectured: $f \in Lip_M^* \alpha$ implies $B_n f \in Lip_M^* \alpha$. Ding–Xuan Zhou showed that the conjecture is false for $0 < \alpha \leq 1$ and he proved that $f \in Lip_M^* \alpha$ implies $B_n f \in Lip_{2M}^* \alpha$, $1 < \alpha \leq 2$ [37, Theorem 4.5]. This result was extended in [21, Theorem 5.2] for the operators of binomial type as follows:

$$\text{if} \quad f \in Lip_M^* \alpha \quad \text{then} \quad L_n^Q f \in Lip_{2M}^* \alpha, \quad 0 < \alpha \leq 1.$$

Examples 3.4. Further on, choosing concrete delta operators Q we reobtain some classical linear positive operators of discrete type.

3.4.1. In the case $Q = D$ we have $Q'^{-2} = I$, consequently (3.3) implies $a_n^{(2)} = 0$ and we get the well–known result concerning the **Bernstein operator** B_n, that is $(B_n e_2)(x) = x^2 + x(1-x)/n$, see e.g. [16, pages 5–6].

3.4.2. If $Q = A_a$ with its basic sequence $\tilde{a}$ (see 2.2.3 and 1.3.3) then $A_a' = E^a(I + aD)$ (Theorem 2.12.(ii)). Assuming that the parameter a is non positive and depends on n, $a := -t_n$ one obtains the **Cheney–Sharma operators** named G_n^* [17], see also the monograph [3, Eq.(5.3.16)]. If the sequence $(nt_n)_{n \geq 1}$ converges to zero then $\lim_{n\to\infty} \|G_n^* f - f\|_\infty = 0$ for every $f \in C([0,1])$. Also a positive integer n_0 exists such that

$$|(G_n^* f)(x) - f(x)| \leq 18\omega_1\left(f; \sqrt{x(1-x)/n}\right) \quad \text{for every} \quad n \geq n_0 \quad \text{and} \quad x \in [0,1].$$

3.4.3. If $Q = \dfrac{1}{\alpha}\nabla_\alpha$, $\alpha \neq 0$, then $Q'^{-2} = E^{2\alpha}$. The basic polynomials will be $p_n(x) = (x + (n-1)\alpha)^{[n,\alpha]}$, see 2.4. In this case L_n^Q becomes **Stancu operator** [32] denoted by $P_n^{[\alpha]}$,

$$(P_n^{[\alpha]} f)(x) = \sum_{k=0}^{n} w_{n,k}(x;\alpha) f\left(\frac{k}{n}\right) \quad \text{where} \quad w_{n,k}(x;\alpha) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},$$

α being a parameter which may depend only on the natural number n. By using (3.3) we have

$$a_n^{(2)} = \frac{\alpha(n-1)}{n(1+\alpha)} \quad \text{and} \quad (P_n^{[\alpha]} e_2)(x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right),$$

in accordance with [32, Lemma 4.1]. If $f \in C([0,1])$ and $0 \leq \alpha = \alpha(n) \to 0$ as $n \to \infty$, then the sequence $(P_n^{[\alpha]} f)_{n \geq 1}$ converges to f uniformly on $[0,1]$ and

$$|(P_n^{[\alpha]} f)(x) - f(x)| \leq \frac{3}{2}\omega_1\left(f; \sqrt{(1+\alpha n)/(n+n\alpha)}\right),$$

see D.D. Stancu's paper [32, Th.4.1 and Th.5.1] or the monograph [3; *5.2.7*].

The operators defined by (3.1) have been generalized in different ways. We present some of them.

3.4.4. In [33] D.D. Stancu and M.R. Occorsio replaced the system of nodes $\{k/n : k = \overline{0,n}\}$ by the following $\{(k+\gamma)/(n+\delta) : k = \overline{0,n}\}$ where γ, δ are parameters satisfying the relations $0 \le \gamma \le \delta$. In the same paper the authors construct another binomial type operator $Q_n^{\alpha,\beta,\gamma,\delta}$ depending on four real parameters, defined for any function $f \in C([0,1])$ by the formula

$$(Q_n^{\alpha,\beta,\gamma,\delta} f)(x) = \sum_{k=0}^{n} q_{n,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{n+\delta}\right),$$

where

$$(1+\alpha+n\beta)^{[n-1,-\alpha]} q_{n,k}^{\alpha,\beta}(x) =$$

$$= \binom{n}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(n-k)\beta)^{[n-1-k,-\alpha]}$$

and $\alpha \ge 0$, $\beta \ge 0$, $\delta \ge \gamma \ge 0$.

In the case $\beta = \gamma = \delta = 0$ one reobtains the Stancu operator $P_n^{[\alpha]}$, see 3.4.3.

If the parameters α and β depend on n such that $\alpha = \alpha(n) \to 0$, $n\beta(n) \to 0$ as $n \to \infty$, the authors proved that $\lim_{n\to\infty} Q_n^{\alpha,\beta,\gamma,\delta} f = f$ uniformly on the interval $[0,1]$ ([33, Theorem 5.1]).

3.2 Modified Operators of binomial type

We now proceed to illustrate some further properties of the binomial sequences. Taking into account the notations in Theorem 2.9., let $Q \in \mathcal{L}_\delta$, $p = (p_n)_{n\ge 0}$, $Q = \phi(D)$, $\phi^{-1} = \varphi$.

For $\varphi(t) = c_1 t + c_2 t^2 + \dots$ we assume $c_1 > 0$ and $c_j \ge 0$, $j \ge 2$, the role of these positivity conditions were revealed by Lemma 3.1. Also $p_n(n) \ne 0$, $n \in \mathbb{N}$, are required. In [17, Th.2.9 and Th.2.10] A. Lupaş proved new inequalities between the terms of the binomial sequences p. For any $x > 0$ one has

(i) $0 < c_1 \dfrac{p_{n-1}(x)}{x} \le (Q'^{-2} p_{n-2})(x) \le \dfrac{p_n(x)}{x^2}$, for any $n \ge 2$;

(ii) $\dfrac{1}{n} \le \rho_n(Q) < 1$ where $\rho_n(Q) := 1 - \dfrac{n(n-1)}{p_n(n)} (Q'^{-2} p_{n-2})(n)$. (3.4)

Following Lupaş we can define the operators $\tilde{L}_n^Q : C([0,1]) \to C([0,1])$,

$$(\tilde{L}_n^Q f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n-nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \tag{3.5}$$

We denote by $\mathcal{B}$ all sequences of linear positive operators $(\tilde{L}_n^Q)_{n\geq 1}$ defined as in (3.5). Also let $\mathbf{K}_s$ be the set of all f, $f \in C([0,1])$, which are non-concave of s^{th} order on $[0,1]$, $s=-1,0,1,\ldots$ This means that for all systems $x_1,x_2,\ldots,x_{s+2}$ of distinct points from $E=[0,1]$, one has

$$[x_1,x_2,\ldots,x_{s+2};f]\geq 0.$$

It should be mentioned that the notion of s–convexity on an arbitrary set E is due to T. Popoviciu, see [25, page 13].

The following properties of $\tilde{L}_n^Q$ were emphasized (see also [18]).

Theorem 3.5. *Suppose that $\tilde{L}_n^Q\in\mathcal{B}$ and $\rho_n(Q)$ is defined by* (3.4).

(i) $\tilde{L}_n(\mathbf{K}_s)\subseteq\mathbf{K}_s$, $s=-1,0,1\ldots$

(ii) *For $f\in\mathbf{K}_1$, $\min\limits_{\tilde{L}_n^Q\in\mathcal{B}}\|\tilde{L}_n^Q f-f\|_\infty = \|B_nf-f\|_\infty$, where $B_n=\tilde{L}_n^D$ is the Bernstein operator.*

(iii) $\tilde{L}_n^Q e_i=e_i$, $i\in\{0,1\}$ *and* $\tilde{L}_n^Q e_2=e_2+(e_1-e_2)\rho_n(Q)$.

(iv) *For any $n\geq 2$ and $f\in C([0,1])$ we have*

$$\|\tilde{L}_n^Q f-f\|_\infty\leq\frac{5}{4}\omega_1\left(f;\sqrt{\rho_n(Q)}\right),\quad \|\tilde{L}_n^Q f-f\|_\infty\leq\frac{9}{8}\omega_2\left(f;\sqrt{\rho_n(Q)}\right).$$

(v) *(A Voronovskaja type formula) Let x_0 be in $[0,1]$ such that $f''(x_0)$ exists and let the polynomials $d_{k,n}\in\Pi_k^*$ be defined by*

$$d_{k,n}(x)=\frac{1}{p_n(n)}\sum_{j=0}^{n-k}\binom{n-k}{j}p_{j+k}(nx)p_{n-k-j}(n-nx)\ .$$

If $\lim\limits_{n\to\infty} n\rho_n(Q)=\overline{\rho}$, $\overline{\rho}>0$, and the condition

$$\lim_{n\to\infty} n^3\sum_{k=0}^{4}(k-nx)^4\sum_{j=k}^{4}(-1)^{j-k}\binom{j}{k}d_{j,n}(x)=0$$

is satisfied, then

$$\lim_{n\to\infty} n(f(x_0)-(\tilde{L}_n^Q f)(x_0))=-\frac{\overline{\rho}x_0(1-x_0)}{2}f''(x_0).$$

Some concrete examples of $\tilde{L}_n^Q$ operators are given below.

Examples 3.6.

3.6.1. If $f_L(t)$ is the indicator of Laguerre operator then we consider the delta operator L^- having its indicator $f_L(-t)$, in other words $L^- = \dfrac{D}{D+I}$. From $\varphi(t) = t(1-t)^{-1}$ we remark that the positivity condition is satisfied. Following 2.14. the basic sequence for L^- is $(l_n)_{n\geq 0}$ where $l_0(x) = 1$, $l_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}x^k$, $n \geq 1$, and one obtains

$$\rho_n(L^-) = -\frac{3n+2}{n^2} + \frac{2}{n^2}\frac{l_{n+1}(n)}{l_n(n)}, \quad [18, \text{pages } 64\text{–}65].$$

Since $\rho_n(L^-) < 3/n$, the sequence $(\tilde{L}_n^{L^-})_{n\geq 1}$ converges to the identity operator on the whole space $C([0,1])$.

3.6.2. Choosing $Q = T$ with its basic sequence $(t_n)_{n\geq 0}$ (see 2.2.6 and 1.3.2) we have $T'^{-2} = (I+D)^2$ and applying Theorem 2.13.(iv) we get $t_n = (X + XD)(t_{n-1})$ consequently

$$T'^{-2}t_{n-2} = \frac{t_n - t_{n-1}}{e_2} \quad \text{and} \quad \rho_n(T) = \frac{1}{n} + \frac{n-1}{n}\frac{t_{n-1}(n)}{t_n(n)} < \frac{2}{n},$$

where $n \geq 2$. The sequence $(\tilde{L}_n^T)_{n\geq 1}$ converges to the identity operator on the space $C([0,1])$.

In [2] we modified the operator L_n^Q defined by (3.1) into an integral form which approximate any integrable function, that is

$$(K_n^Q f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad f \in L_1([0,1]), \tag{3.6}$$

where $p_{n,k}(x) = \dfrac{1}{p_n(1)}\dbinom{n}{k}p_k(x)p_{n-k}(1-x)$.

For $Q = D$, K_n^D becomes the n^{th} Kantorovich operator, see e.g. [3; *5.3.7*]. The degree of approximation was estimated by using the r^{th} order modulus of smoothness of f measured in $L_p([0,1])$–spaces, $p \geq 1$, that is

$$\omega_r(f;t)_p := \sup_{0<|h|\leq t} \|\Delta_h^r f\|_p, \quad f \in L_p([0,1]), \quad t > 0.$$

For more details we refer to [9].

Theorem 3.7. *Let $(K_n^Q)_{n\geq 1}$ be defined by (3.6) such that the notations in Theorem 3.2 are preserved and the positivity condition is fulfilled. Let $f \in L_p([0,1])$, $p \geq 1$.*

(i) *If $\lim_{n\to\infty} q_n = 0$ then $\lim_{n\to\infty} \|K_n^Q f - f\|_p = 0$, where $q_n = r_{n-2}(1)/p_n(1)$, $n \geq 2$, $r_n(x)$ is generated by (3.2) and $\|\cdot\|_p$ represents the usual norm in $L_p([0,1])$ spaces.*

(ii) *If $r \geq 3$ is an integer, for sufficiently large n one has*

$$\|K_n^Q f - f\|_p \leq 2(|q_n| + 2/n)\|f\|_p + C_{p,r}\omega_r(f; |q_n|^{1/r} + (2/n)^{1/r})_p,$$

where $C_{p,r}$ is a constant independent of f and n.

3.3 Exponential–type operators

In the sequel we need a brief introduction about the exponential–type operators. Let us consider an interval $J = (a, b)$ where $-\infty \leq a < b \leq \infty$ and let $W_\lambda(t, \cdot)$ be the density function of a random variable $X_{\lambda,t}$, $(\lambda, t) \in (0, \infty) \times J$. Since we don't exclude variables with discrete distribution, for $W_\lambda(t, \cdot)$ we use the term of *generalized function*, see [5, page 206]. However, such a function is nonnegative and verifies the normalization condition

$$\int_{\mathbb{R}} W_\lambda(t, u)du = 1. \tag{3.7}$$

Let p be a nonnegative analytic function on J. An exponential operator is a positive linear integral operator

$$(S_\lambda f)(t) = \int_{\mathbb{R}} W_\lambda(t, u) f(u)du, \tag{3.8}$$

whose kernel W_λ satisfies the partial differential equation

$$\frac{\partial}{\partial t} W_\lambda(t, u) = \lambda \frac{u - t}{p(t)} W_\lambda(t, u). \tag{3.9}$$

The exponential operators with $p \in \Pi_2$ were studied by C.P. May [20]. In this class we recover some classical linear positive operators, such as: the Gauss–Weierstrass operator ($\lambda = 1$, $J = \mathbb{R}$, $p(t) = 1$), the Szász operator ($\lambda \in \mathbb{N}$, $J = (0, \infty)$, $p(t) = t$), the Baskakov operator ($\lambda \in \mathbb{N}$, $J = (0, \infty)$, $p(t) = t + t^2$); all of them are presented in the monograph [3; *5.2.9, 5.3.9, 5.2.6*]. Knowing that $S_\lambda e_j = e_j$, $j \in \{0, 1\}$, and $S_\lambda e_2 = e_2 + p/\lambda$ [34, Proposition 3.1] it is obvious that the exponential operators are approximation operators, that is $S_\lambda f \to f$ as $\lambda \to \infty$, for certain spaces of functions f (in general, bounded by some growth–test function).

In [13] Ismail and May identified the exponential operator (3.8) as the bilateral Laplace transform

$$(S_\lambda f)(t) = \int_{\mathbb{R}} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)} d\theta\right) C_\lambda(u) f(u) du, \tag{3.10}$$

for some $c \in J$, the normalization (3.7) becoming

$$\int_{\mathbb{R}} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)} d\theta\right) C_\lambda(u) du = 1. \tag{3.11}$$

Notice two facts:

(i) The operator S_λ of (3.10) is independent of c, cf. [12, Lemma 2.3].

(ii) If $C_{\lambda,1}, C_{\lambda,2}$ are generalized functions such that, for some $c \in J$, the condition (3.11) holds then they coincide, cf. [36, page 69]. So, there is at most one generalized function C_λ satisfying (3.11).

As regards the interval J we assume that $a > -\infty$ and the function $1/p$ has a simple pole at $z = a$. By a linear change of variables in t and u we can take $a = 0$ and

$$1/p(z) = 1/z + h(z), \tag{3.12}$$

with h analytic.

We introduce the functions ξ and η as follows

$$\xi(t) = \frac{1}{c} \exp\left(\int_c^t h(\theta) d\theta\right), \quad \eta(\xi(t)) = t - c + \int_c^t \theta h(\theta) d\theta.$$

Furthermore $\eta'(0) \neq 0$ and one can choose c in order to make $\eta(0) \neq 0$. Also we define the map φ, $\varphi(\xi) = \eta(\xi) - \eta(0)$ and in according to our relation (2.5) let's denote $\psi = (\psi_n)_{n\geq 0}$ the sequence of basic polynomials generated by φ. We have

$$\exp\{\lambda(\eta(\xi) - \eta(0))\} = \sum_{n\geq 0} \psi_n(\lambda) \frac{\xi^n}{n!}.$$

Under the above assumptions and notations we can infer

(i) The generalized function C_λ satisfying (3.11) is a sum of delta functions as follows ([12, Theorem 3.3])

$$C_\lambda(u) = e^{\lambda\eta(0)} \sum_{n\geq 0} \frac{\psi_n(\lambda)}{n!} \delta(n - \lambda u).$$

Substituting this relation in (3.10) the explicit form of the operator S_λ becomes

(ii) $$(S_\lambda f)(t) = \exp\{-\lambda(\eta(\xi)-\eta(0))\}\sum_{k\geq 0}\psi_k(\lambda)\frac{\xi^k}{k!}f\left(\frac{k}{\lambda}\right). \tag{3.13}$$

A converse to this statement is given below

(iii) Every basic set of polynomials $\psi = (\psi_n(\lambda))_{n\geq 0}$ of binomial type generates an integral operator (3.8) with $a = 0$ and p of the form (3.12). The integral operator is given explicitely in (3.13).

Indeed, setting $t = \xi\frac{d\eta}{d\xi}(\xi) = \xi\frac{d\varphi}{d\xi}(\xi)$, we define the generalized function $W_\lambda(t,\cdot)$ by

$$W_\lambda(t,u) = \xi^{\lambda u}\exp\{-\lambda(\eta(\xi)-\eta(0))\}\sum_{k=0}^{\infty}\psi_k(\lambda)\frac{\delta(k-\lambda u)}{k!}.$$

We easily check the requirement (3.9)

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial \xi}\frac{d\xi}{dt} = \frac{d\xi}{dt}\left(\frac{\lambda u}{\xi} - \frac{\lambda d\eta}{d\xi}(\xi)\right)W = \lambda\frac{u-t}{\xi}\frac{d\xi}{dt}W,$$

with $p(t) = \xi\frac{dt}{d\xi}$. Also $\frac{1}{p(t)}$ has a simple pole at $t = 0$ and $\left.\frac{dp(t)}{dt}\right|_{t=0} = 1$. So the proof is complete.

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