Korean J. Comput. & Appl. Math. Vol. **9***(2002), No. 2, pp. 361 - 372*

ON APPROXIMATION PROPERTIES OF BALÁZS-SZABADOS OPERATORS AND THEIR KANTOROVICH EXTENSION

Octavian Agratini

Abstract. In this paper we deal with a sequence of positive linear operators $R_n^{[\beta]}$ approximating functions on the unbounded interval $[0,\infty)$ which were firstly used by K. Balázs and J. Szabados. We give pointwise estimates in the framework of polynomial weighted function spaces. Also we establish a Voronovskaja type theorem in the same weighted spaces for $K_n^{[\beta]}$ operators, representing the integral generalization in Kantorovich sense of the $R_n^{[\beta]}$.

AMS Mathematics Subject Classification : 41A36, 41A60.
Key words and phrases : weighted space, linear and positive operator, Kantorovich-type operator, Voronovskaja-type theorem.

1. Introduction

Let f be a real valued function defined on $[0,\infty)$. In [5] K. Balázs considered Bernstein type rational functions belonging to f defined as follows

$$(R_n f)(x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right), \quad x \ge 0,\ n \in \mathbb{N}, \tag{1}$$

where a_n and b_n are suitably chosen real numbers independent of x. Under the conditions:

(i) $a_n = b_n/n$, $b_n \to \infty$ $(n \to \infty)$,

(ii) f is continuous on $[0,\infty)$ and satisfies $f(x) = \mathcal{O}(e^{\alpha x})$ $(x \to \infty)$, for some real number α,

the sequence $(R_n f)_{n\ge 1}$ becomes an approximation process, that is $R_n f$ converges to f uniformly on any compact subinterval of the semi-axis $[0,\infty)$. A special attention was given to the particular case

$$a_n = n^{\beta-1}, \quad b_n = n^{\beta}, \quad n = 1,2,3,\ldots, \quad 0 < \beta < 1, \tag{2}$$

Received July 12, 2001. Revised November 6, 2001 .

where the above operators were denoted by $R_n^{[\beta]}$.

Here are some of the most outstanding results concerning these operators obtained so far. K. Balázs and J. Szabados [6] gave weighted estimates and investigated the uniform convergence of $R_n^{[\beta]}$ operators.

These operators were also used to construct a quadrature formula with positive coefficients and equidistant nodes to approximate certain improper integrals. V. Totik [11] settled the saturation properties of $R_n^{[\beta]}f$, proved a general convergence theorem for R_n-like rational functions and obtained a Voronovskaja-type result. In 1989 Biancamaria Della Vecchia [9] proved some preservation properties and asymptotic relations for $R_n^{[\beta]}$, $0<\beta<1$, operators. A recent paper is due to Ulrich Abel and B. Della Vecchia [1] who obtained the complete asymptotic expansion for $R_n^{[\beta]}$ operators as n tends to infinity, all coefficients being calculated explicitly.

Considering that $b_n=n$, $a_n>0$, in [3] we modified the R_n operators into integral form operators by replacing $f(k/b_n)$ with an integral mean of $f(x)$ over a small interval named $I_{n,k}:=\left[\frac{k}{na_n},\frac{k+1}{na_n}\right]$, as follows

$$(K_nf)(x)=\frac{na_n}{(1+na_n)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\int_{I_{n,k}}f(t)dt,\quad x\geq 0,\quad n\in\mathbb{N},\tag{3}$$

where $f\in\mathcal{M}_{loc}(\mathbb{R}_+)$, the class of all measurable functions on $\mathbb{R}_+=[0,\infty)$ and bounded on every compact subinterval of $\mathbb{R}_+$.

Choosing $a_n=n^{\beta-1}$ where $0<\beta<1$ is fixed, the operator defined by (3) becomes the integral analogue (named $K_n^{[\beta]}$) of $R_n^{[\beta]}$. More precisely, we have

$$(K_n^{[\beta]}f)(x)=n^\beta\sum_{k=0}^{n}r_{n,k}(x)\int_{kn^{-\beta}}^{(k+1)n^{-\beta}}f(t)dt,\quad x\geq 0,\tag{4}$$

where

$$r_{n,k}(x)=(1+n^{\beta-1}x)^{-n}\binom{n}{k}(n^{\beta-1}x)^k,\quad k=\overline{0,n}.\tag{5}$$

Since $\sum_{k=0}^{n}r_{n,k}(x)=1$ we obtain

$$(R_n^{[\beta]}e_0)(x)=(K_n^{[\beta]}e_0)(x)=1,\quad x\geq 0,\quad n\in\mathbb{N},\tag{6}$$

where $e_0(x)=1$.

Regarding to the K_n operators, in the mentioned paper [3] some approximation properties are revealed establishing the rate of pointwise convergence for various classes of functions such as continuous functions, local $Lip\alpha$ $(0<\alpha\leq 1)$ functions and locally bounded having the one-sided limits at a given $x_0>0$.

The present paper focuses on two approaches. Firstly, we discuss some new approximation properties of the $R_n^{[\beta]}$ operators in the setting of polynomial weighted function spaces. Secondly, we give a Voronoskaja-type theorem for the Balázs-Kantorovich $K_n^{[\beta]}$ operators. These results are presented in Section 3 and respectively Section 4. In Section 2 we recall some classical notations used throughout the paper and establish some results concerning both with $R_n^{[\beta]}$ and $K_n^{[\beta]}$ operators.

2. Definitions and preliminary results

As usual, the symbol $\mathbb{N}_0$ is used to denote the set $\{0\}\cup\mathbb{N}$. We denote by $C(\mathbb{R}_+)$ the vector space of all-valued continuous functions on $[0,\infty)$. $C_B(\mathbb{R}_+)$ denotes the Banach lattice of all real-valued bounded continuous functions on $[0,\infty)$ endowed with the natural order and the sup-norm $\|\cdot\|_\infty$, $\|f\|_\infty := \sup\limits_{t\geq 0}|f(t)|$. For every $N \in \mathbb{N}_0$ we shall denote by e_N the monomials defined as $e_0(t) = 1$ and $e_N(t) = t^N$ $(N \geq 1)$, where $t \in \mathbb{R}_+$. Moreover, we consider a net (w_N) of weighted functions as follows

$$w_0(t) = 1, \quad w_N(t) = (1 + t^N)^{-1}, \quad N \in \mathbb{N}, \quad t \in \mathbb{R}_+.$$

$C_N(\mathbb{R}_+)$ stands for the Banach space consisting of all functions $f \in C(\mathbb{R}_+)$ such that $w_N f \in C_B(\mathbb{R}_+)$, endowed with the norm

$$\|f\|_N := \sup_{t\geq 0} w_N(t)|f(t)|, \quad f \in C_N(\mathbb{R}_+). \tag{7}$$

Obviously, we have $\|f\|_N \leq \|f\|_\infty$ for all $N \in \mathbb{N}_0$ and $f \in C_B(\mathbb{R}_+)$. The study in such spaces is welcome because corresponding to the unbounded interval the approximated functions are allowed to be unbounded, provided they have polynomial growth at infinity.

In order to prove our results we need the functions $\psi_{x,j}$, $j \in \mathbb{N}$, defined by $\psi_{x,j}(t) = (t-x)^j$, $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, and the j-th central moments for $R_n^{[\beta]}$ respectively $K_n^{[\beta]}$ operators, that means

$$\mathcal{M}_{n,j}(x) := (R_n^{[\beta]}\psi_{x,j})(x), \quad \widetilde{\mathcal{M}}_{n,j}(x) := (K_n^{[\beta]}\psi_{x,j})(x), \quad x \in \mathbb{R}_+.$$

Based on previous results, see [5;*Lemma 2.1*] and [3;*Lemma 1*], we deduce

$$\mathcal{M}_{n,1}(x) = -\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}, \quad \mathcal{M}_{n,2}(x) = \frac{(n^{2\beta-2}x^3 + n^{-\beta})x}{(1+n^{\beta-1}x)^2} \tag{8}$$

and respectively

$$\begin{aligned} \widetilde{\mathcal{M}}_{n,1}(x) &= -\frac{n^{\beta-1}x}{1+n^{\beta-1}x} + \frac{1}{2n^\beta}, \\ \widetilde{\mathcal{M}}_{n,2}(x) &= \frac{(n^{2\beta-1}x^2 - 1)x^2}{n(1+n^{\beta-1}x)^2} + \frac{(1-n^{\beta-1}x)x}{n^\beta(1+n^{\beta-1}x)} + \frac{1}{3n^{2\beta}}. \end{aligned} \tag{9}$$

Examining the relations (8) we can state

Lemma 1. *If* $\lambda := \min\{1-\beta, \beta\}$ *then for every* $x \in \mathbb{R}_+$ *the following inequalities*

$$|\mathcal{M}_{n,1}(x)| \le \frac{1+x^3}{2n^\lambda}\mu_\beta(n,x), \quad \mathcal{M}_{n,2}(x) \le \frac{1+x^3}{n^\lambda}\mu_\beta^2(n,x) \tag{10}$$

hold true, where $\mu_\beta(n,x) := \sqrt{x}(1+n^{\beta-1}x)^{-1}$.

At this point we recall some facts about the Stirling numbers $s(n,m)$ of the second kind. These numbers [2;*24.1.4*] verify the recurrences

$$s(n+1,m) = ms(n,m) + s(n,m-1), \quad n \ge m \ge 1,$$

and the following identitites:

$$s(n,1) = s(n,n) = 1, \quad s(n,0) = \delta_{0,n}, \quad s(n,n-1) = \binom{n}{2}$$

hold true as well. Actually, $s(n,m)$ is the number of ways of partitioning a set of n elements into m non-empty subsets.
Setting $T_N(n,y) := \sum_{k=0}^{n}\binom{n}{k}k^N y^k$, $N \ge 1$, $n \ge 1$, we obtain $y\frac{d}{dy}T_N(n,y) = T_{N+1}(n,y)$ and by using the Stirling numbers of the second kind we can write

$$T_N(n,y) = \sum_{i=0}^{N} s(N,i)y^i\frac{d^i}{dy^i}[(1+y)^n]. \tag{11}$$

Now we shall prove

Lemma 2. *For each* $N \in \mathbb{N}_0$ *there exists a constant* K_N *such that*

$$w_N(x)R_n^{[\beta]}\left(\frac{1}{w_N}, x\right) \le K_N, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Proof. If $N=0$ then we have $w_0(x)R_n^{[\beta]}\left(\frac{1}{w_0}, x\right) = R_n^{[\beta]}(e_0, x) = 1 := K_0$, see (6).
If $N=1$ then we have $w_1(x)R_n^{[\beta]}\left(\frac{1}{w_1}, x\right) = (1+x)^{-1}R_n^{[\beta]}(e_0+e_1, x) \le 1 := K_1$.

Further, we consider $N \ge 2$. Using (11) we obtain

$$R_n^{[\beta]}(e_N, x) = \frac{n^{-N\beta}}{(1+n^{\beta-1})^N}T_N(n, n^{\beta-1}x)$$

$$= \frac{n^{-N\beta}}{(1+n^{\beta-1}x)^N}\sum_{i=0}^{N} s(N,i)(n^{\beta-1}x)^i n(n-1)\dots(n-i+1)(1+n^{\beta-1}x)^{n-i}$$

$$= n^{-N\beta}\sum_{i=1}^{N} s(N,i)\langle n\rangle_i\left(\frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right)^i \le \sum_{i=1}^{N} s(N,i)x^i,$$

where $\langle n\rangle_i$ stands for the lower-factorials.

Using the elementary inequality $\sup\limits_{x\geq 0}(x+x^2+\cdots+x^N)(1+x^N)^{-1}\leq N$ we get

$$w_N(x)R_n^{[\beta]}(e_N,x)\leq N\Delta_N,\quad \Delta_N:=\max_{i=\overline{1,N}}s(N,i), \tag{12}$$

and consequently

$$w_N(x)R_n^{[\beta]}\left(\frac{1}{w_N},x\right)=w_N(x)(1+R_N^{[\beta]}(e_N,x))\leq 1+N\Delta_N:=K_N.\ \square$$

Lemma 3. *For every $N\in\mathbb{N}_0$ a constant K_N exists such that for all f belongs to $C_N(\mathbb{R}_+)$ one gets*

$$\|R_n^{[\beta]}f\|_N\leq K_N\|f\|_N,\quad n\in\mathbb{N}, \tag{13}$$

where $\|\cdot\|_N$ is defined by (7).

Proof. Taking into account Lemma 2, for every $f\in C_N(\mathbb{R}_+)$ and $x\geq 0$ we have

$$\begin{aligned}|w_N(x)(R_n^{[\beta]}f)(x)| &\leq w_N(x)R_n^{[\beta]}\left(w_N|f|\frac{1}{w_N},x\right)\\ &\leq w_N(x)\|f\|_N R_n^{[\beta]}\left(\frac{1}{w_N},x\right)\leq K_N\|f\|_N,\end{aligned}$$

and this implies (13). $\square$

Lemma 4. *For every $N\in\mathbb{N}$, there exists a positive constant $\widetilde{K}_N(x)$, independent of N, such that the relation*

$$w_N(x)R_n^{[\beta]}(\psi_{x,2}/w_N,x)\leq \widetilde{K}_N(x)n^{-\gamma},\quad x\in\mathbb{R}_+,$$

holds, where $\gamma=\min\{\beta/2,1-\beta\}$.

Proof. When $x=0$ the relation is evident. When $N=0$ the assertion results from (10). Now we suppose that $N\geq 1$ and $x>0$. Since $\psi_{x,2}/w_N=\psi_{x,2}+e_N\psi_{x,2}$, by using Cauchy inequality we get

$$\begin{aligned}&w_N(x)R_n^{[\beta]}(\psi_{x,2}/w_N,x)\leq w_N(x)\mathcal{M}_{n,2}(x)+\\ &+\{w_N(x)(R_n^{[\beta]}e_{2N})(x)\}^{1/2}\{w_N(x)(R_n^{[\beta]}\psi_{x,4})(x)\}^{1/2}.\end{aligned} \tag{14}$$

Lemma 1 and respectively the relation (12) guarantee

$$w_N(x)\mathcal{M}_{n,2}(x)\leq x(1+x^3)n^{-\lambda},\ \{w_N(x)(R_n^{[\beta]}e_{2N})(x)\}^{1/2}\leq\sqrt{2N\Delta_{2N}}. \tag{15}$$

On the other hand, by using the result due to U. Abel and B. Della Vecchia [1;*Lemma 3.4*] we can infer that there exists $c_N(x)>0$, independent of n, such that

$$\{w_N(x)(R_n^{[\beta]}\psi_{x,4})(x)\}^{1/2}\leq c_N(x)n^{-2\gamma},\quad \gamma:=\min\{\beta/2,1-\beta\}. \tag{16}$$

It is clear that $\gamma \le \min\{\beta, 1-\beta\} = \lambda$. Inserting both (15) and (16) into (14) we have

$$\begin{aligned} w_N(x) R_n^{[\beta]}(\psi_{x,2}/w_N, x) &\le x(1+x^3)n^{-\gamma} + \sqrt{2N\Delta_{2N}}c_N(x)n^{-2\gamma} \\ &\le \frac{x + x^4 + \sqrt{2N\Delta_{2N}}c_N(x)}{n^\gamma}, \end{aligned}$$

and this completes the proof. □

Coming back to $K_n^{[\beta]}$ operators, by simple calculations on relations (9) we can state

Lemma 5. *Let $x > 0$ be fix. The central moments of first and second order of $K_n^{[\beta]}$ operators verify the following relations*
(i) if $0 < \beta < 1/2$ then $\lim_{n\to\infty} n^\beta \widetilde{\mathcal{M}}_{n,j}(x) = \begin{cases} 1/2, & j = 1; \\ x, & j = 2; \end{cases}$
(ii) if $\beta = 1/2$ then $\lim_{n\to\infty} \sqrt{n}\widetilde{\mathcal{M}}_{n,j}(x) = \begin{cases} -x + 1/2, & j = 1; \\ x, & j = 2; \end{cases}$
(iii) if $1/2 < \beta < 1$ then $\lim_{n\to\infty} n^{1-\beta} \widetilde{\mathcal{M}}_{n,j}(x) = \begin{cases} -x, & j = 1; \\ 0, & j = 2. \end{cases}$

Based on [1;*Lemma 3.4*], see also (16), and noticing that

$$(K_n^{[\beta]}\psi_{x,4})(x) = \frac{1}{5}\sum_{j=0}^{4} \binom{5}{j} n^{-(4-j)\beta} (R_n^{[\beta]}\psi_{x,j})(x),$$

we present another technical result.

Lemma 6. *For every $N \in \mathbb{N}$, $\widetilde{c}_N(x)$ exists, positive and independent of n, such that the inequality*

$$(K_n^{[\beta]}\psi_{x,4})(x) \le \widetilde{c}_N(x) n^{-4\gamma}, \quad x \in \mathbb{R}_+,$$

holds, where $\gamma = \min\{\beta/2, 1-\beta\}$.

Now we are going to evaluate $K_n^{[\beta]} e_N$. Using (6) and (12) we have

$$\begin{aligned} (K_n^{[\beta]} e_N)(x) &= \frac{1}{N+1}\sum_{j=0}^{N}\binom{N+1}{j}\sum_{k=0}^{n} r_{n,k}(x)\left(\frac{k}{n^\beta}\right)^j \left(\frac{1}{n^\beta}\right)^{N-j} \\ &\le \frac{1}{N+1}\sum_{j=0}^{N}\binom{N+1}{j}\frac{j\Delta_j}{w_N(x)} = \frac{1}{w_N(x)}\sum_{j=1}^{N}\binom{N}{j-1}\Delta_j. \end{aligned}$$

Following the same method as in the proof of Lemma 2, we formulate

Lemma 7. *For each $N \in \mathbb{N}_0$ there exists a constant c_N such that*

$$w_N(x) R_n^{[\beta]}\left(\frac{1}{w_N}, x\right) \le c_N, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

On the light of the previous relations it turns out that c_N could be expressed by using Stirling numbers of the second kind such as

$$1 + \sum_{j=1}^{N} \binom{N}{j-1} \max_{i=\overline{1,j}} s(j,i).$$

Also we mention that Lemma 7 implies $\left\| K_n^{[\beta]}\left(\frac{1}{w_N}, \cdot\right) \right\|_N \le c_N$.

3. Estimates for $R_n^{[\beta]}$ operators in the spaces $C_N(\mathbb{R}_+)$

Strongly based on the work of Michael Becker [7], further results incorporate the w_N weights into our approximation statements regarding with the $R_n^{[\beta]}$ operators.

Further on we consider

$$C_N^2(\mathbb{R}_+) := \{h : \; h, h', h'' \in C_N(\mathbb{R}_+)\}.$$

Theorem 1. *For every $N \in \mathbb{N}$ and $g \in C_N^2(\mathbb{R}_+)$ the following inequality*

$$w_N(x) |(R_n^{[\beta]} g)(x) - g(x)| \le n^{-\gamma} (A(x) \|g'\|_N + B(x) \|g''\|_N), \quad x \in \mathbb{R}_+,$$

holds, where $\gamma = \min\{\beta/2, 1-\beta\}$ and

$$A(x) = \frac{\sqrt{x}}{2}(1 + x^3), \quad B(x) = \frac{1}{2}(x^4 + x + \widetilde{K}_N(x)). \tag{17}$$

Proof. For $x = 0$ the inequality is obvious. Let $x > 0$. Fixing $N \in \mathbb{N}$ and $g \in C_N^2(\mathbb{R}_+)$ we can write for all $t \ge 0$ and $x > 0$

$$g(t) - g(x) = (t-x) g'(x) + \int_x^t (t-u) g''(u) du. \tag{18}$$

Let $J_{x,t}$ be the interval having endpoints in x and t, thus the length of $J_{x,t}$ is $|x-t|$. Since $\sup\limits_{u \in J_{x,t}} w_N(u) |g''(u)| \le \|g''\|_N$ we have

$$\begin{aligned} \left| \int_{J_{x,t}} (t-u) g''(u) du \right| &\le \|g''\|_N \left| \int_{J_{x,t}} \frac{|t-u|}{w_N(u)} du \right| \\ &\le \frac{\|g''\|_N}{2} (t-x)^2 \left(\frac{1}{w_N(x)} + \frac{1}{w_N(t)} \right) \end{aligned} \tag{19}$$

Taking into account (6), as well as the relations (18), (19) and (8), we get from Lemmas 1 and 4

$$w_N(x)|(R_n^{[\beta]}g)(x)-g(x)| = w_N(x)|R_n^{[\beta]}(g-g(x),x)|$$

$$\le w_N(x)|g'(x)||\mathcal{M}_{n,1}(x)| + \frac{1}{2}\|g''\|_N\left(\mathcal{M}_{n,2}(x)+w_N(x)R_n^{[\beta]}\left(\frac{\psi_{x,2}}{w_N},x\right)\right)$$

$$\le \frac{1+x^3}{2n^\lambda}\mu_\beta(n,x)\|g'\|_N + \frac{1}{2}\left(\frac{1+x^3}{n^\lambda}\mu_\beta^2(n,x)+\frac{\widetilde{K}_N(x)}{n^\gamma}\right)\|g''\|_N.$$

Because $\mu_\beta(n,x)\le\sqrt{x}$ and $\lambda\ge\gamma$, we obtain the desired result. □

Following Becker's paper [7] we consider the modified modulus of smoothness

$$\omega_{2,N}(f,\delta) := \sup_{0<h\le\delta}\|\Delta_h^2 f\|_N,\quad f\in C_N(\mathbb{R}_+),\quad \delta>0, \tag{20}$$

where $\Delta_h^2 f(x) := f(x+2h)-2f(x+h)+f(x)$ represents the second forward difference associated to f and x with step $h>0$.

The central result of this section can be stated as follows.

Theorem 2. *Let $f\in C_N(\mathbb{R}_+)$. The operators defined by (1) and (2) have the following property*

$$w_N(x)|(R_n^{[\beta]}f)(x)-f(x)| \le (\widehat{K}_N + n^{-\gamma/2})\omega_{2,N}(f, C(x)n^{-\gamma/2}),\quad x\ge 0,$$

where $\widehat{K}_N$ is a constant depending only on N, $C(x)=\max\left\{4A(x), 3\sqrt{B(x)}\right\}$ and $A(x)$, $B(x)$, are defined by (17).

Proof. Note that for $x=0$ the relation becomes trivial. Further, for any $h>0$ we introduce the modified Steklov means by

$$f_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}(2f(x+s+t)-f(x+2s+2t))dsdt.$$

From this formula and the definition of Δ_h^2 operator one has

$$\begin{aligned}
f(t)-f_h(x) &= \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(x)dsdt,\\
f_h'(x) &= \frac{2}{h}(\Delta_{t_0}^2 f(x)-\Delta_{t_0+h/2}^2 f(x)),\quad 0<t_0\le\frac{h}{2},\\
f_h''(x) &= \frac{1}{h^2}(8\Delta_{h/2}^2 f(x)-\Delta_h^2 f(x))
\end{aligned}$$

and hence, according to (20),

$$\|f-f_h\|_N \le \omega_{2,N}(f,h), \tag{21}$$

$$\|f_h'\|_N \le 4h^{-1}\omega_{2,N}(f,h),\quad \|f_h''\|_N \le 9h^{-2}\omega_{2,N}(f,h). \tag{22}$$

We can write

$$w_N(x)|(R_n^{[\beta]}f)(x)-f(x)| \le w_N(x)\{|R_n^{[\beta]}(f-f_h,x)|+$$

$$+|(R_n^{[\beta]}f_h)(x)-f_h(x)|+|f_h(x)-f(x)|\}.$$

Now we estimate the terms on the right side separately. Using (13), (21) and (8) we obtain

$$w_N(x)|R_n^{[\beta]}(f-f_h,x)| \le K_N\omega_{2,N}(f,h), \quad w_N(x)|f_h(x)-f(x)| \le \omega_{2,N}(f,h).$$

Using Theorem 1 ($f_h \in C_N^2(\mathbb{R}_+)$) and (22) we get

$$\begin{aligned} w_N(x)|(R_n^{[\beta]}f_h)(x)-f_h(x)| &\le n^{-\gamma}\left(\frac{4A(x)}{h}+\frac{9B(x)}{h^2}\right)\omega_{2,N}(f,h) \\ &\le (n^{-\gamma/2}+1)\omega_{2,N}(f,C(x)n^{-\gamma/2}). \end{aligned}$$

We choose $h=C(x)n^{-\gamma/2}$ where $C(x)=\max\left\{4A(x),3\sqrt{B(x)}\right\}$. Collecting the above inequalities we obtain the claimed result, where $\widehat{K}_N$ could be chosen as K_N+2. $\square$

Remarks. 1) Theorem 2 in particular implies that for each $f\in C_N(\mathbb{R}_+)$ and $x\ge 0$, $\lim_{n\to\infty} w_N(x)|(R^{[\beta]}f)(x)-f(x)|=0$ holds true.

2) For any fixed $\alpha\in(0,2]$, we consider the modified Lipschitz class corresponding to $\omega_{2,N}$ as follows

$$Lip_{2,N}\alpha := \{f\in C_N(\mathbb{R}_+):\ \omega_{2,N}(f,\delta)=\mathcal{O}(\delta^\alpha),\ \delta\to 0^+\}.$$

If a function h belongs to $Lip_{2,N}\alpha$, $0<\alpha\le 2$, then a constant $M_f\ge 0$ exists such that $\omega_{2,N}(h,\delta)\le M_h\delta^\alpha$. Consequently, Theorem 2 states that

$$w_N(x)|(R_n^{[\beta]}h)(x)-h(x)| \le c(h,N)\left(\frac{C^{1/\gamma}(x)}{\sqrt{n}}\right)^{\alpha\gamma},$$

where $c(h,N)\le M_h(\widehat{K}_N+1)$.

Incorporating the weights w_N both into the above approximation and into the definition of $Lip_{2,N}\alpha$ we can take an important step forward which leads to the global estimates $\|C^{-\alpha}(x)((R_n^{[\beta]}h)(x)-h(x))\|_N=\mathcal{O}(n^{-\alpha\gamma/2})$.

3) Such an approach of the estimates in the weighted spaces is familiar to several recent papers, of which I quote the extensive paper of I. Carbone [8] in which the author deals with some sequences of positive linear operators introduced earlier in a joint paper with Francesco Altomare [4] and representing a generalization of Szász-Mirakjan operators.

4. A Voronovskaja type theorem for $K_n^{[\beta]}$ operators

At first we prove a new property of $K_n^{[\beta]}$ operators.

Theorem 3. *Let $x \in \mathbb{R}_+$ and $N \in \mathbb{N}_0$ be fixed. Let $g_x \in C_N(\mathbb{R}_+)$ such that*

$$\lim_{t\to x} g_x(t) = 0. \tag{23}$$

Then $\lim\limits_{t\to x}(K_n^{[\beta]} g_x)(x) = 0$ holds true.

Proof. We fix $\varepsilon > 0$ and consider the constant c_N given by Lemma 7. From (23) and the fact that $g_x \in C_N(\mathbb{R}_+)$ we deduce there exist the positive constants $\delta = \delta(\varepsilon, c_N)$ and M such that

$$w_N(t)|g_x(t)| < \frac{\varepsilon}{2c_N} \text{ for } t \in \mathbb{R}_+,\ |t-x| < \delta, \tag{24}$$

$$w_N(t)|g_x(t)| \le M \text{ for all } t \in \mathbb{R}_+. \tag{25}$$

Setting $U_{n,1} \cup U_{n,2} := \{1 \le k \le n : \ |kn^{-\beta} - x| < \delta\} \cup \{1 \le k \le n : |kn^{-\beta} - x| \ge \delta\}$, we can write

$$w_N(x)|(R_n^{[\beta]} g_x)(x)| \le w_N(x) n^\beta \Big\{ \sum_{k\in U_{n,1}} r_{n,k}(x) \int_{kn^{-\beta}}^{(k+1)n^{-\beta}} |g_x(t)|dt +$$

$$+ \sum_{k\in U_{n,2}} r_{n,k}(x) \int_{kn^{-\beta}}^{(k+1)n^{-\beta}} |g_x(t)|dt \Big\} := S_{n,1} + S_{n,2}.$$

From (24) and Lemma 7 we obtain

$$\begin{aligned} S_{n,1} &\le \frac{\varepsilon}{2c_N} w_N(x) n^\beta \sum_{k\in U_{n,1}} r_{n,k}(x) \int_{kn^{-\beta}}^{(k+1)n^{-\beta}} \frac{dt}{w_N(t)} \\ &\le \frac{\varepsilon}{2c_N} w_N(x) K_n^{[\beta]}\left(\frac{1}{w_N}, x\right) \le \frac{\varepsilon}{2}. \end{aligned}$$

Since $\displaystyle\int_{kn^{-\beta}}^{(k+1)n^{-\beta}} \frac{dt}{w_N(t)} \le \frac{1}{n^\beta w_N((k+1)n^{-\beta})} \le \frac{2^N}{n^\beta w_N(kn^{-\beta})}$, from (25) we have

$$\begin{aligned} S_{n,2} &\le M w_N(x) n^\beta \sum_{k\in U_{n,2}} r_{n,k}(x) \int_{kn^{-\beta}}^{(k+1)n^{-\beta}} \frac{dt}{w_N(t)} \\ &\le 2^N M w_N(x) \sum_{k\in U_{n,2}} r_{n,k}(x) \frac{1}{w_N(kn^{-\beta})}. \end{aligned}$$

If $k \in U_{n,2}$ then $1 \le \delta^{-2}(kn^{-\beta} - x)^2$ and by using Lemma 4 we continue

$$\begin{aligned} S_{n,2} &\le 2^N M\delta^{-2} w_N(x) \sum_{k=0}^{n} r_{n,k}(x) \left(\frac{k}{n^\beta} - x\right)^2 \frac{1}{w_N(kn^{-\beta})} \\ &\le 2^N M\delta^{-2} w_N(x) R_n^{[\beta]}(\psi_{x,2}/w_N, x) \le 2^N M\delta^{-2} \widetilde{K}_N(x) n^{-\gamma}. \end{aligned}$$

For the fixed positive numbers N, M, δ, x there exists a natural number n_0 such that $S_{n,2} < \varepsilon/2$, for all $n \ge n_0$.

Consequently, gathering the above relations we obtain

$$w_N(x)|(R_n^{[\beta]} g_x)(x)| < \varepsilon \text{ for all } n \ge n_0,$$

and the proof of our theorem is complete. □

At this moment we can state the main theorem of this section giving an answer in a certain direction to the speed with which $K_n^{[\beta]} f$ tends to f.

Theorem 4. *Let $N \in \mathbb{N}_0$ and $f \in C_N^2(\mathbb{R}_+)$. Then for every $x \in \mathbb{R}_+$ the following identities hold true*

(i) if $0 < \beta < 1/2$ then $\lim_{n\to\infty} n^\beta \Delta(K_n^{[\beta]} f, x) = \frac{1}{2}(f'(x) + xf''(x))$;

(ii) if $\beta = 1/2$ then $\lim_{n\to\infty} \sqrt{n}\Delta(K_n^{[\beta]} f, x) = \frac{1}{2}((1-2x)f'(x) + xf''(x))$;

(iii) if $1/2 < \beta < 1$ then $\lim_{n\to\infty} n^{1-\beta}\Delta(K_n^{[\beta]} f, x) = -xf'(x)$.

Here $\Delta(K_n^{[\beta]} f, x) := (K_n^{[\beta]} f)(x) - f(x)$.

Proof. We rely on the approach of P.C. Sikkema [10]. By the Taylor's formula for $f \in C_N^2(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$ we have

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2 \frac{f''(x)}{2} + \varphi(x,t)(t-x)^2, \quad t \ge 0,$$

where $\varphi(x, \cdot) \in C_N(\mathbb{R}_+)$ and $\lim_{t\to x} \varphi(t,x) = 0$.

Since $K_N^{[\beta]}$ is a linear positive operator, (6) and the above identity imply

$$(K_n^{[\beta]} f)(x) = f(x) + f'(x)\widetilde{\mathcal{M}}_{n,1}(x) + \frac{f''(x)}{2}\widetilde{\mathcal{M}}_{n,2}(x) + K_n^{[\beta]}(\psi_{x,2}\varphi(x,\cdot), x). \quad (26)$$

Further, using the Cauchy inequality we get

$$|K_n^{[\beta]}(\psi_{x,2}\varphi(x,\cdot), x)| \le \{K_n^{[\beta]}(\psi_{x,4}, x)\}^{1/2} \{K_n^{[\beta]}(\varphi^2(x,\cdot), x)\}^{1/2}.$$

Lemma 6 guarantees that $n^\lambda \{K_n^{[\beta]}(\psi_{x,4}, x)\}^{1/2} = \mathcal{O}(1)$ $(n \to \infty)$.

If in Theorem 3 we choose $g_x := \varphi^2(x,\cdot) \in C_{2N}(\mathbb{R}_+)$ then the above relations imply $\lim_{n\to\infty} K_n^{[\beta]}(\psi_{x,2}\varphi(x,\cdot), x) = 0$, and taking into account Lemma 5, the identity (26) leads us to the claimed result. □

Remark. Introducing $F_f := e_1 f'$ for any $f \in C_N^2(\mathbb{R}_+)$, the conclusion of Theorem 4 can be written as

$$\lim_{n\to\infty} n^\lambda((K_n^{[\beta]} f)(x) - f(x)) = a_\beta \frac{dF_f}{dx}(x) + b_\beta F_f(x),$$

where $\lambda = \min\{\beta, 1-\beta\}$, $0 < \beta < 1$, and the constants a_β, b_β have the following values

$$a_\beta = \begin{cases} 1/2, & \beta \in (0,1/2], \\ 0, & \beta \in (1/2,1), \end{cases} \qquad b_\beta = \begin{cases} 0, & \beta \in (0,1/2), \\ -1, & \beta \in [1/2,1). \end{cases}$$

References

1. U. Abel, B. Della Vecchia, *Asymptotic approximation by the operators of K. Balázs and Szabados*, Acta Sci. Math. (Szeged), **66**(2000), 137-145.
2. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, 55 Issued June, 1964.
3. O. Agratini, *An approximation process of Kantorovich type*, Mathematical Notes (Miskolc), **2**(2001), 1, 3-10.
4. F. Altomare, I. Carbone, *On a new sequence of positive linear operators on unbounded intervals*, Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl., **40**(1996), 23-36.
5. K. Balázs, *Approximation by Bernstein type rational functions*, Acta Math. Acad. Sci. Hungar., **26**(1975), f.1-2, 123-134.
6. C. Balázs, J. Szabados, *Approximation by Bernstein type rational functions. II*, Acta Math. Acad. Sci. Hungar., **40**(1982), 331-337.
7. M. Becker, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J., **27**(1978), n.1, 127-142.
8. I. Carbone, *Approximation properties of some sequences of positive linear operators on weighted function spaces*, Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl., **52**(1998), 305-336.
9. B. Della Vecchia, *On some preservation and asymptotic relations of a rational operator.* Facta Universitatis (Niš), **4**(1989), 57-62.
10. P.C. Sikkema, *On some linear positive operators*, Indagationes Mathematicae, **32**(1970), 4, 327-337.
11. V. Totik, *Saturation for Bernstein type rational functions*, Acta Math. Hung., **43**(1984), f.3-4, 219-250.

Octavian Agratini graduated the Faculty of Mathematics and Computer Science of Babeş-Bolyai University, Cluj-Napoca, Romania, in 1981 and obtained the Ph. D. under the guidance of Professor D.D. Stancu, in 1995. Between 1991-1996 he was a Lecturer and since 1996 he has been an Associate Professor at the Babeş-Bolyai University. He has obtained two research grants: in Heidelberg (Germany, 1996) and in Bari (Italy, 2000), the latest under the guidance of Professor Francesco Altomare. He is a member of the A.M.S. and his interest field is the approximation of functions by linear operators.

Faculty of Mathematics and Computer Science, "Babeş-Bolyai" University, str. Kogălniceanu, 1, 3400 Cluj-Napoca, Romania
`e-mail : agratini@math.ubbcluj.ro`